Minimal surfaces in holographic entanglement entropy

Anthony Nguyen

Abstract

We give an exposition of a few papers related to the asymptotic Plateau problem and the holographic entanglement entropy problem from quantum gravity. Also, we sketch an argument for a mountain pass theorem in hyperbolic space after we describe its connections to a certain computational complexity problem in black hole information theory. The notes collected are from an undergraduate research project at Cornell advised by Professor Xin Zhou.

Contents

1 Introduction

Minimal surfaces are surfaces which locally minimize area. In addition to being purely of mathematical interest, such surfaces find an abundance of applications in physics. One such modern problem is the usage of minimal surfaces and their “quantum” generalizations towards the black hole information paradox. In this paper we give an exposition of the theory of minimal surfaces relevant to understanding these current applications in quantum gravity.

Much of the motivation follows from the following classical Plateau problem: given any prescribed simple closed curve γ^{k-1} find a minimal surface M^k whose boundary is $\partial M = \gamma$. The classical theorem was proved in the 1930's by Douglas and Rado. For us it is particularly interesting to consider minimal surfaces in hyperbolic space $\mathbb{H}^n$. More precisely, the asymptotic Plateau problem concerns the existence of minimal hypersurfaces in hyperbolic manifolds given a prescribed boundary at infinity. Anderson resolved the existence theory in the case of hyperbolic space in [1] and [2] using techniques from geometric measure theory. Later in [3], Alexakis and Mazzeo derive a formula for the area of such minimal surfaces after subtraction of some asymptotic infinite term in the case of hyperbolic 3-manifolds.

The outline of the paper is as follows. We first review the prerequisite geometric measure theory and hyperbolic geometry necessary, and then proceed towards an exposition of some of the proofs found in Anderson's papers. Next we will put into context the problem of the holographic entanglement entropy from quantum gravity. In particular we will outline the development of the black hole information paradox posed by Hawking as well was current posed resolutions by physicists such as Ryu, Takayanagi, Hubeny, Wall, Engelhardt and many others. We also discuss the complexity theory of obtaining information from black holes as well as its geometric obstructions, termed the "Python's Lunch" in physics. We describe how such physical results are due to Mountain Pass theorems in hyperbolic space. Finally we present an original argument for a mountain pass theorem in hyperbolic space.

2 Preliminaries

2.1 Geometric Measure Theory

Geometric Measure Theory (GMT) provides a useful framework in extending tools from differential geometry to sets in $\mathbb{R}^n$ which are not necessarily smooth. Here we summarize without proof important definition and results used in the remainder of the notes. As we shall see, GMT is a natural tool for studying minimal surfaces due to some very powerful compactness and regularity results. The definitions and results of this section are taken from [4] and [1].

Definition 2.1. A set $M \subset \mathbb{R}^{n+l}$ is said to be **countably** $n-$ **rectifiable** if

$$M = M_0 \cup \left(\cup_{j=1}^{\infty} F_j(M_j)\right)$$

where $M_j \subset \mathbb{R}^n$, M_0 a set of Hausdorff measure $\mathscr{H}(M_0) = 0$, and $F_j : M_j \to \mathbb{R}^{n+l}$ are Lipschitz functions.

Lemma 2.1. *M is countably n-rectifiable if and only if $M \subset \cup_{j=0}^{\infty} N_j$, with $\mathscr{H}(N_0) = 0$ and each N_j with $j \geq 1$, is an n-dimensional embedded C^1 submanifold of $\mathbb{R}^{n+l}$*

In view the preceding lemma, one may interpret rectifiable sets as a generalization of differential manifolds to include cusps and singularities. We also have the analogous notion of approximate tangent spaces defined almost everywhere, and in fact these tangent spaces characterize n-rectifiable sets. More precisely,

Definition 2.2. if M is a $\mathscr{H}^n$-measurable subset of $\mathbb{R}^{n+l}$ with finite Hausdorff measure when restricted to compact sets K, then we define an n-dimensional subspace $T_x M$ of $\mathbb{R}^{n+l}$ as the **approximate tangent space** for M at x satisfying

$$\lim_{\lambda \downarrow 0} \int_{\eta_{x,\lambda}(M)} f(y) d\mathscr{H}^n(y) = \int_{T_x M} f(y) d\mathscr{H}^n(y)$$

for all $f \in C_c^0(\mathbb{R}^{n+l})$, where $\eta_{x,\lambda} : \mathbb{R}^{n+l} \to \mathbb{R}^{n+l}$ is defined by $\eta_{x,\lambda}(y) = \lambda^{-1}(y - x)$ for $x, y \in \mathbb{R}^{n+l}$ and $\lambda > 0$.

Theorem 2.2. *Suppose M is $\mathscr{H}^n$-measurable with $\mathscr{H}^n(M \cap K)$ finite for each compact subset K. Then M is countably $n-$rectifiable if and only if the approximate tangent space $T_x M$ exists for $\mathscr{H}^n$-almost everywhere $x \in M$.*

Now given an oriented C^∞ Riemannian manifold (N, g) denote $\Omega^p(N)$ as the space of p-forms on N.

Definition 2.3. The **space of p-currents** $\Omega_p(N)$ is the space of continuous linear functionals on $\Omega^p(N)$ with the weak topology.

For oriented, precompact, finite submanifolds M^p with finite p-dim volume, we have a corresponding p-current $[M]$ defined by

$$[M](\omega) = \int_M \omega$$

for $\omega \in \Omega^p(N)$.

Definition 2.4. A **rectifiable p-current** is a convergent sum of currents

$$\sum_j [M_j]$$

where $\{M_j\}_1^\infty$ is a collection of mutually disjoint $p-$rectifiable sets. Denote by $\mathscr{R}(N)$ the **space of rectifiable $p-$currents**.

Definition 2.5. For $\omega \in \Omega^p(N)$ define the **comass** by

$$\|w\| := \sup \{|\langle w, \xi \rangle|\}$$

where ξ is a unit, simple p-vector. The **mass norm** on $\mathscr{R}_p(N)$ is then given by

$$\underline{M}(T) := \sup \left\{ T(w) : \sup_{x \in N} \|\omega(x)\| \leq 1 \right\}$$

where we note the mass of a current represents the weighted area of a generalized surface.

Definition 2.6. For $T \in \mathscr{R}_p(N)$, the **total variation measure** of T is defined as

$$\|T\|(A) = \underline{M}(T \llcorner A)$$

where $T \llcorner A(\omega) = T(\chi_A \cdot \omega)$ denotes restriction of T to A via the characteristic function.

Definition 2.7. The **support** of $T = \sum_j [M_j]$ is the closure of its constituent rectifiable sets,

$$\operatorname{supp} T = \overline{\sum_{j=1}^{\infty} M_j}$$

Now if the boundary ∂M of M is rectifiable, then by Stoke's theorem it defines a current given by

$$[\partial M](\omega) = \int_{\partial M} \omega = \int_M d\omega = [M](d\omega)$$

Then we define the space of **integral p-currents** $\mathscr{I}_p(N)$ to be the set of $T \in \mathscr{R}_p(N)$ such that $\partial T \in \mathscr{R}_{p-1}(N)$. As a trivial remark note that a rectifiable current need not be an integral current; consider for example any bounded surface whose boundary has infinitely many wiggles $\iff$ infinite length boundary.

We can also define $\mathscr{I}_p^{loc}$ the **subset of locally integral $p-$currents** as the currents T such that $\forall x \in N \exists \mathscr{T} \in \mathscr{I}_p$ of compact support with $x \neq \operatorname{supp}(T - \mathscr{T})$.

What follows are the major theorems of GMT.

Theorem 2.3 (Federer-Fleming Compactness Theorem)**.** *If $K \subset N$ is a compact set and $c \in \mathbb{R}_+$, then the set*

$$S = \{T \in \mathscr{I}_*(N) : \operatorname{supp} T \subset K, \underline{M}(T) + \underline{M}(\partial T) \leq c\}$$

is compact in the weak topology. That is, for any sequence $\{T_j\} \subset S$ there is an integer multiplicity $T \in S$ and a subsequence $\{T_{j'}\}$ such that $T_{j'} \rightharpoonup T$ in N.

Proof. See [4, p. 149] □

Theorem 2.4 (Regularity)**.** *An $(n-1)-$dimensional, area-minimizing rectifiable current $T \in \mathbb{R}^n$ is a smooth, embedded manifold on the interior except for a singular set of Hausdroff measure at most $n-8$. If $n=8$, singularities are isolated points.*

Proof. See [5, 5.3.16] □

Theorem 2.5 (Homology)**.** *The boundary operator $\partial : \Omega_{p+1} \to \Omega_p$ given by $(\partial T)(\omega) = T(d\omega)$ defines a chain complex on $\mathscr{I}_*(N)$, and we have that the homology associated to $\{\mathscr{I}_n(N), \partial\}$ is isomorphic to singular homology on N.*

$$H_*(\mathscr{I}_*(N)) \simeq H_*(N, \mathbb{Z})$$

Theorem 2.6 (Isoperimetric)**.** *Suppose $T \in \Omega_{n-1}(\mathbb{R}^{n+l})$ is integer multiplicity, $n \geq 2$ and $\operatorname{supp} T$ is compact with $\partial T = 0$. Then there is an integer multiplicity current $R \in \Omega_n(\mathbb{R}^{n+l})$ with $\operatorname{supp} R$ compact and $\partial R = T$ and $\underline{M}(R)^{\frac{n-1}{n}} < c(n,k)\underline{M}(T)$*

By the lower-semi-continuity of the mass function with respect to weak convergence, there is a solution to the Plateau problem in the space of integral currents via the above compactness theorem.

Definition 2.8. A current $T \in \mathscr{I}_p^{loc}(N)$ is **stationary** if for any compact $K \subset N$, and all vector fields V (with flows ϕ_t^V) with compact support and $\operatorname{supp}(V) \subset K$ we have

$$\begin{aligned} \frac{d}{dt}\underline{M}\left((\phi_t^V)_*(T \llcorner K)\right) &= \sum_{i=1}^{p} \int_n \left\langle \nabla_{e_j} V, e_j \right\rangle d\|T\| = 0 \\ &= \int_N \operatorname{div} V d\|T\| = 0 \end{aligned}$$

Definition 2.9. A current $T \in \mathscr{I}_p^{loc}(N)$ is **absolutely area-minimizing** if $\forall K \subset N$ one has

$$\underline{M}(T \llcorner K) \leq \underline{M}(\mathscr{T})$$

for any $\mathscr{T} \in \mathscr{I}_p(N)$ with $\partial(T \llcorner K) = \partial \mathscr{T}$

2.2 Hyperbolic Geometry

See for [6] and [7] for more details regarding the following.

Definition 2.10. The n-dimensional hyperbolic space $\mathbb{H}^n$ is the unique simply connected, Riemannian manifold of constant sectional curvature equal to -1.

There are many useful models of hyperbolic space. The first of which that we will primarily need to use is the **Poincaré ball model** which identifies $\mathbb{H}^n$ with a unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ via a conformal equivalence under the metric

$$ds^2 = (4\frac{dx_1^2 + \cdots + dx_n^2}{1 - (x_1^2 + \cdots + x_n^2))^2}$$

The second useful model is the **Poincaré half-space model** which identifies $\mathbb{H}^n$ with the upper half-space $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ via the metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}$$

and both metrics are complete. From the point of view of the hyperbolic metrics, the models have no boundary. Hyperbolic straight lines extend infinitely long, although hyperbolic distances are represented by increasingly smaller euclidean distances towards the edge in each model. However it is useful to consider the edge of these models as a sort of "conformal boundary". The sphere $\partial\mathbb{B}^n = \mathbb{S}^{n-1}(\infty)$ is called the (boundary) sphere at infinity in the first model. The corresponding asymptotic boundary in the second model is the plane $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n = 0\}$. The isometries of $\mathbb{H}^n$ extend to $\partial\mathbb{H}^n$ as conformal automorphisms, for example Möbius transformations in the case $n = 3$.

The geodesics in the ball model are arcs of circles which intersect the boundary sphere orthogonally. The geodesics in the half-space model are represented half-circles whose origin is on the $x_n = 0$-plane and straight vertical rays normal to the $x_n = 0$-plane.

3 Minimal Surfaces in Hyperbolic Space

We give an exposition of some proofs of [1] and [2] for the existence theory for complete minimal varieties and complete minimal surfaces in hyperbolic space $\mathbb{H}^n$. We will primarily work in the Poincaré ball model of hyperbolic space. To begin, we first prove the Monotonicity formula which will be used in many subsequent arguments.

Theorem 3.1 (Monotonicity Formula)**.** *Let φ be a stationary integral $p-$ current in a Riemannian manifold N^n of sectional curvature $\kappa_N \le -a^2 \le 0$. Let* $\mathrm{vol}(B^p(-a^2, r))$ *denote the $p-$dimensional volume of a geodesic ball of radius r in the simply connected complete Riemannian manifold of constant sectional curvature $-a^2$. Denote B_r a geodesic r-ball in N^n. Then*

$$\psi(r) = \frac{\underline{M}(\varphi \llcorner B_r)}{\mathrm{vol}(B^p(-a^2, r))}$$

is monotone non-decreasing in r for all geodesic r-balls contained in N.

Proof. Denote by B_r a geodesic r-ball centered at some point $p \in N$. Let $r : N \to \mathbb{R}$ be the distance function from p so that its level sets are B_r. Denote by $T_y(\varphi)$ the approximate tangent space at $y \in \operatorname{supp}\varphi$ and let $\{e_j\}_1^p$ be an orthonormal basis.

Let E be a smooth vector field of compact support $K \subset N$. In particular we take vector fields of the form

$$E = f(r) \cdot \chi_r \cdot (r \operatorname{grad} r)$$

where $f(r)$ is any smooth function. Furthermore χ_r is a smooth approximation to the characteristic function $\chi_{[0,r]}$ in the following sense: $\chi_{[0,r]}$ is continuous on $(0, r)$ and $\mathbb{R} \setminus [0, r]$ so write as a convolution with a smooth bump function. The 1st variational formula reads

$$\begin{aligned}\sum_{j=1}^{p} \int_K \langle \nabla_{e_j} E, e_j \rangle d\|\varphi\| &= \sum_j \int_K \langle \nabla_{e_j} f \cdot \chi_r \cdot (r \operatorname{grad} r), e_j \rangle d\|\varphi\| \\ &= \sum_j \int_K f \cdot \chi_r \langle \nabla_{e_j} r \operatorname{grad} r, e_k \rangle d\|\varphi\| \\ &= \int_K f \cdot \chi_r \operatorname{div}(r \operatorname{grad} r) d\|\varphi\| \\ &= \int_{\partial K} f \cdot \chi_r \langle r \operatorname{grad} r, N \rangle - \int_K \langle \operatorname{grad}(f \cdot \chi_r), r \operatorname{grad} r \rangle\end{aligned}$$

□

the boundary integral obviously vanishes, so for $x = r \operatorname{grad} r$ and $x^T =$ the projection of x on $T\varphi$ we have

$$r \operatorname{grad} r = \frac{\|x^T\|}{x} r \operatorname{grad} r = \frac{\|x^T\|^2}{\|x\|}$$

and consequently

$$\sum_{j=1}^{p} \int_K \nabla_{e_j} E, e_j \rangle d\|\varphi\| = -\int_K \operatorname{grad}(f \cdot \chi_r) \cdot \frac{\|x^T\|^2}{\|x\|} d\|\varphi\|$$

For any $q \in B_r \cap \operatorname{supp} \varphi$ consider the unique geodesic sphere S_q^{n-1} centered at p and intersecting q. Choose a basis for $T_q(\varphi)$ such that

$$\{e_i\}_1^{p-1} \in T_q\varphi \cap T_q S^{n-1}$$

and decompose e_p into its radial and tangential parts

$$e_p = \langle e_p, \operatorname{grad} r \rangle \cdot \operatorname{grad} r + \langle e_p, e_p^\perp \rangle \cdot e_p^\perp$$

with $e_p^\perp$ the direction vector along the projection of e_p on $T_q S^{n-1}$. Denote B the 2nd fundamental form of the geodesic r-sphere centered at p. Since $\operatorname{grad} r$ is a normal vector field on r-sphere centered at p, and e_i is perpendicular to the radius vector of r-sphere, we have for $i \leq p-1$ that

$$\langle \nabla_{e_i} r \operatorname{grad} r, e_i \rangle = r \langle \nabla_{e_i} \operatorname{grad} r, e_i \rangle = r B(e_i, e_i)$$

Now let J_i be the unique Jacobi field such that $J_i(p) = 0, J_i(q) = e_i$ and let J denote J_i for $i \leq p-1$. If I is the index form then we know

$$\begin{aligned} B(e_i, e_i) = I(J, J) &= \int_0^r \langle \nabla_T J, \nabla_T J \rangle - \langle R(T, J)J, T \rangle \\ &= \int_0^r \|\nabla_T J\|^2 - K_N |J \wedge T|^2 \\ &\geq \int_0^r \|\nabla_T J\|^2 + a^2 \|J\|^2 \end{aligned}$$

The fundamental inequality of the index form in M_{-a^2} says that

$$\int_0^r \|\nabla_T J\|^2 + a^2 \|J\|^2 \geq I^a(J_a, J_a)$$

where I^a, J_a are the respective index forms and Jacobi fields in M_{-a^2}. In fact we have $J_a = h \cdot E$ for E a parallel vector field and

$$h(t) = \frac{\sinh(at)}{\sinh(ar)}$$

We have $I(J, J) \geq I^a(J_a, J_a)$. Since $I^a(J_a, J_a = \langle \nabla_T J_a, J_a \rangle$,

$$\begin{aligned} I(J, J) &\geq \langle \nabla_T h \cdot E, h \cdot E \rangle = \langle h \nabla_T E + h'(t), h \cdot E \rangle \\ &= \left\langle \frac{a \cosh(at)}{\sinh(ar)} \cdot E, \frac{\sinh(at)}{\sinh(ar)} \cdot E \right\rangle \\ &= \frac{a \cosh(at)}{\sinh(ar)} \cdot \frac{\sinh(at)}{\sinh(ar)} \langle E, E \rangle \\ &\geq a \coth(ar) \end{aligned}$$

and additionally,

$$\begin{aligned} \langle \nabla_{e_p} r \operatorname{grad} r, e_p \rangle = \langle e_p, \operatorname{grad} r \rangle \langle \nabla_{e_p} r \operatorname{grad} r, \operatorname{grad} r \rangle + \\ \langle e_p, e_p^\perp \rangle \langle \nabla_{e_p} r \operatorname{grad} r, e_p^\perp \rangle \\ \geq \langle e_p, \operatorname{grad} r \rangle^2 + \langle e_p, e_p^\perp \rangle^2 \cdot ar \cdot \coth(ar) \end{aligned}$$

giving

$$\begin{aligned} \sum_{i=1}^p \langle \nabla_{e_i} r \operatorname{grad} r, e_i \rangle \geq \langle e_p, \operatorname{grad} r \rangle^2 + \langle e_p, e_p^\perp \rangle^2 \cdot ar \cdot \coth(ar) \\ + (p-1) ar \cdot \coth(ar) \end{aligned}$$

Let $Q = 1 + (p-1) ar \cdot \coth(ar)$. Then

$$\sum_{i=1}^p \langle \nabla_{e_i} r \operatorname{grad} r, e_i \rangle \geq Q$$

and by the previous formula we get

$$\int f \cdot \chi_r \cdot Q d\|\varphi\| \leq -\int [\chi_r' \cdot f + f' \cdot \chi_r] \frac{\|x^T\|^2}{\|x\|} d\|\varphi\|$$

using the $\|x^T\|^2/\|x\| = r\|\mathrm{grad}\, r^T\|^2$ we rewrite as

$$\int \chi_r \cdot [fq + f'r]] d\|\varphi\| + \int \chi_r \cdot f'r \left[\|\mathrm{grad}\, r^T\|^2 - 1\right] d\|\varphi\|$$
$$\leq \int \chi_r' f r \|\mathrm{grad}\, r^T\| \cdot d\|\varphi\|$$

Now choose f

$$f = \frac{\mathrm{vol}(B^p(-a^2, r))}{r \cdot \mathrm{vol}(S^{p-1}(a^2, r))} = \frac{\int_0^r \sinh^{p-1}(ar)}{r \sinh^{p-1}(ar)}$$

a straightforward calculation gives us

$$f'(r) = \frac{1}{r} - \frac{1}{r^2} \int_0^r \sinh^{p-1}(ar) \left[\sinh^{1-p}(ar) + (ap - a) r \cosh(ar) \sinh^{-p}(ar)\right]$$

from which we readily see that $f \cdot Q + f'r \equiv 1$ and $f'(r) \leq 0$ for $r > 0$. The choice of f gives $\int \chi_r f' r [\left\|\mathrm{grad}^T r\right\|^2 - 1] d\|\varphi\| \geq 0$ which thus yields by the above

$$\int \chi_r d\|\varphi\| \leq -\int \chi_r' \cdot f \cdot r \cdot d\|\varphi\|$$

where χ_r' is a surface delta function ≤ 1. Choose χ_r to be sufficiently close approximation to the characteristic function $\chi_{[0,r]}$ we have

$$\underline{M}(\varphi \llcorner B_r) \leq r \cdot f \cdot \underline{M}'(\varphi \llcorner B_r)$$

and accordingly that

$$\frac{\underline{M}(\varphi \llcorner B_r)}{\int_0^r \sinh^{p-1}(at) dt}$$

is monotone non-decreasing.

3.1 Existence of Area-Minimizing Currents

Now the compactification of $\mathbb{H}^n$ in view of the Poincaré ball model is given by

$$\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \mathbb{S}^{n-1}(\infty)$$

so we definite the **asymptotic boundary** (or **ideal boundary**) S of a locally integral p-current Σ in $\mathbb{H}^n$ by

$$S = \overline{\operatorname{supp}\Sigma} \cap \mathbb{S}^{n-1}(\infty)$$

Our focus in this section is the following Theorem:

Theorem 3.2. *Let $M^{p-1} \to \mathbb{S}^{n-1}(\infty)$ be a smooth immersion of a closed oriented manifold into the boundary at infinity. Then there is a complete area-minimizing locally integral $p-$current Σ in $\mathbb{H}^n$ with asymptotic boundary M^{p-1}.*

Firstly, let us establish some preliminary definitions and lemmas. Given a subset S of $\overline{\mathbb{H}^n}$, define the **convex hull** $\operatorname{Conv}(S)$ of S to be the intersection of all hyperbolic half spaces containing S. We have

Lemma 3.3. *Let $S \subset \mathbb{S}^{n-1}(\infty)$ be any closed set. Then*

$$\operatorname{Conv}(S) \cap \mathbb{S}^{n-1}(\infty) = S$$

Proof. Proof by contrapositive. Let $U = \mathbb{S}^{n-1}(\infty) - S$. Then U is open, so for any $x \in U$ there is a ball S_x in U centered at x. Let P_x be the totally geodesic hyperplane in $\overline{\mathbb{H}^n}$ orthogonal to $\mathbb{S}^{n-1}(\infty)$ at S_x. Then P_x separates $\operatorname{Conv}(S)$ and x such that they each lie in distinct components of $\overline{\mathbb{H}^n} \setminus P_x$. Hence $x \in \operatorname{Conv}(S) \implies x \in S$. □

Lemma 3.4. *Let Σ be a stationary integral p-current with boundary $\partial\Sigma$. Then*

$$\operatorname{supp}\Sigma \subset \operatorname{Conv}(\operatorname{supp}(\partial\Sigma))$$

Proof. Let P be a hyperplane with half spaces H^+, H^- and suppose that we have $\operatorname{supp}(\partial\Sigma) \subset H^-$. We show $\operatorname{supp}(\Sigma) \subset H^-$. If not, consider $\Sigma \llcorner H^+$, which is a stationary current with

$$\operatorname{supp}(\partial(\Sigma \llcorner H^+)) \subset P$$

By the 1st variational formula, we have

$$\sum_i \int \langle \nabla_{e_j} E, e_j \rangle d\|\Sigma\| = 0$$

for vector fields E of compact support. Let $r(x) = \operatorname{dist}(x, P)$ for $x \in \mathbb{H}^n$, which is obviously convex on $\mathbb{H}^n$. Choosing E by

$$E = \begin{cases} \rho(r) \cdot \operatorname{grad} r & \text{on } H^+ \\ 0 & \text{on } H^- \end{cases}$$

where $|\rho(r) - r| < \varepsilon$ on $\mathrm{supp}(\Sigma \llcorner H^+)$. Then E is smooth and vanishes on $\mathrm{supp}(\partial(\Sigma \llcorner H^+))$, as $\rho(0) = 0$, so it is a variational vector field. Furthermore, we see

$$\begin{aligned}\langle \nabla_{e_j} E, e_j \rangle &= \langle \nabla_{e_j} (\rho(r) \operatorname{grad} r), e_j \rangle \\ &= \langle \rho(r) \nabla_{e_j} \operatorname{grad} r + \rho'(r) \operatorname{grad} r, e_j \rangle \\ &= \rho(r) \left\langle \nabla_{e_j} \operatorname{grad} r, e_j \right\rangle + \rho'(r) \langle \operatorname{grad} r, e_j \rangle\end{aligned}$$

Convexity of r implies that $\operatorname{grad} r$ is monotone (increasing), and we may choose ρ such that $\rho'(r) > 0$, hence

$$\left\langle \nabla_{e_j} E, e_j \right\rangle > 0$$

pointwise on $\mathrm{supp}(\Sigma \llcorner H^+)$. But this contradicts the 1st variational formula for $\Sigma \llcorner H^+$ stationary (violates maximum principle). Since P an arbitary hyperplane, we get the result. □

Proof of Theorem 3.2. Given $M \subset \mathbb{S}^{n-1}(\infty)$, choose an origin $0 \in \mathrm{Conv}(M)$ and retract M smoothly to 0 by a geodesic flow. Let

$$M_t = \{p : p \in \gamma_q(t)\}$$

where γ_q is the normal geodesic from 0 to $q \in M$. Then M_t is a smooth 1-parameter family of immersed manifolds, each contained in the geodesic t-sphere $\mathbb{S}^{n-1}(t)$ centered at 0. For any finite t let Σ_t be a solution to the Plateau problem with boundary M_t. Recall this is justified by lower semi-continuity of the mass function with respect to weak convergence along with Theorem 2.3. We will show that Σ_t has a convergent subsequence to a minimizing, locally integral p-current in $\mathbb{H}^n$ with ideal boundary M, for any sequence t_i going to infinity. Again by Theorem 2.3, it will be necessary and sufficient to find bounds $C_r > c_r > 0$ and $R_0 \in \mathbb{R}$ such that $\forall r \geq R_0$,

$$c_r \leq \underline{M}(\Sigma_t \llcorner B_r) \leq C_r$$

[Existence of C_r]

Since Σ_t is mass-minimizing, we have

$$\underline{M}(\Sigma_t) \leq \mathrm{vol}(C(M_t)) \iff \frac{\underline{M}(\Sigma_t)}{\mathrm{vol}(C(M_t))} \leq 1$$

where $C(M_t) = \{\lambda x \mid x \in M_t; 0 < \lambda \leq 1\}$ is the **cone** on M_t from 0. Also,

$$\mathrm{vol}(C(M_t)) = \alpha \cdot \mathrm{vol}(B^p(t))$$

where $B^p(t)$ is a geodesic t-ball in $\mathbb{H}^n$, and α is given by

$$\alpha = \lim_{r \to 0} \frac{\mathrm{vol}(C(M_t) \cap B_r)}{\mathrm{vol}(B_r)}$$

Here α is independent of t since $C(M_t) \cap B_r = C(M_s) \cap B_r$ whenever $s, t > r$. By the monotonicity result in Theorem 3.1, we have

$$\psi(r) = \frac{\|\Sigma_t\|(B_r)}{\operatorname{vol} B^p(r)}$$

is non-decreasing in r. Fixing a t, for any $r \leq t$ we have

$$\frac{\|\Sigma_t\|(B_r)}{\operatorname{vol} B^p(r)} \leq \frac{\|\Sigma_t\|(B_t)}{\operatorname{vol} B^p(t)} = \frac{\underline{M}(\Sigma_t)}{\frac{1}{\alpha}\operatorname{vol}(C(M_t))} \leq \alpha$$

so that

$$\underline{M}(\Sigma_t \llcorner B_r) \leq \alpha \cdot \operatorname{vol}(B^p(r)).$$

Since $\|\Sigma_t\|(B_s) = \underline{M}(\Sigma_t)$ for all $s \geq t$, we set $C_r \equiv \alpha \cdot \operatorname{vol}(B^p(r))$ as our upper bound.
[Existence of c_r] By Lemma 3.4, we have $\operatorname{supp}(\Sigma_t) \subset \operatorname{Conv}(M_t)$, so consider the behavior of $\operatorname{Conv}(M_t)$ as $t \to \infty$. For a fixed $0 \in \operatorname{Conv}(M_t)$, we have $C(M_t) \subset \operatorname{Conv}(M_t)$, so that $M_s \subset \operatorname{Conv}(M_t)$ for any $s \leq t$. Thus for $s \leq t$,

$$\operatorname{Conv}(M_s) \subset \operatorname{Conv}(M_t)$$

Because $0 \in \operatorname{Conv}(M)$, for $t \to \infty$ we have

$$\bigcup_{t \geq 0}^{\infty} \operatorname{Conv}(M_t) \subset \operatorname{Conv}(M)$$

Recall from Lemma 3.3 that $\operatorname{Conv}(M) \cap \mathbb{S}^{n-1}(\infty) = M$. The support of Σ_t lies in its $\operatorname{Conv}(M_t)$ by previous lemma, hence $\Sigma_t \subset \operatorname{Conv}(M)$ for all t.

Next we prove the existence of a compact set $K \subset \mathbb{H}^n$ such that

$$\operatorname{supp} \Sigma_t \cap K \neq \varnothing$$

for all t. Denote by N a fixed tubular neighborhood of M in $\overline{\mathbb{H}^n} = \overline{B^n}$. A neighborhood deformation retraction in the weak sense implies homotopy equivalence of the inclusion, and it is clear that M_t may not lie in N at t near 0. Hence for all t sufficiently large, the inclusion $M_t \subset N$ is a homotopy equivalence. Suppose by contradiction we have $\operatorname{supp} \Sigma_t \subset N$ for t large. As currents we have $\partial \Sigma_t = M_t$ so that

$$0 = [M_t] \in H_*(\mathscr{I}_*(N))$$

It follows that $[M_t] = 0$ in $H_*(N, \mathbb{Z}) \simeq H_*(M_t, \mathbb{Z})$. A contradiction since the top homology group is always $\mathbb{Z}$, hence M_t is not homologous to zero. This shows

$$\operatorname{supp} \Sigma_t \cap N^c \neq \varnothing \qquad \forall t$$

Since $N^c \cap \operatorname{Conv}(M)$ is compact, we have the existence of K. This follows since $\operatorname{Conv}(M)$ is compact by compactness of $M \subset \mathbb{S}^{n-1}(\infty) \hookrightarrow \mathbb{R}^n$.

To obtain the lower bound c_r, choose R_0 so that $K \subset B(R_0)$ and let $p_t \in \operatorname{supp}(\Sigma_t) \cap B(R_0)$. Then $B(p_t, r - R_0) \subset B(r)$ for $r > R_0$, so that

$$\|\Sigma_t\|(B_r) \geq \|\Sigma_t\|(B(p_t, r - R_0)) \geq \alpha \cdot \operatorname{vol} B^p(r - R_0)$$

with α from Lemma 3.1. Thus we obtain

$$c_r = \operatorname{vol}(B(r - R_0))$$

for a lower bound for $\underline{M}(\Sigma_t \llcorner B_r), r \geq R_0$. We now have bounds

$$0 < c_r \leq \|\Sigma_t\|(B_r) \leq C_r$$

so we can proceed with applying compactness theorem 2.3 and isoperimetric theorem 2.6. Choose a sequence $t_j \to \infty$ and let $\{B_i\}_{i=1}^{\infty}$ be the compact exhaustion of $\mathbb{H}^n$ by balls of radius i. By the above theorems, for any B_i with $i \geq R_0$,

$$\{\Sigma_{t_j} \llcorner B_i\}_{j=1}^{\infty} \text{ has a convergent subsequence.}$$

Choose such a sequence for each i. Taking the diagonal subsequence gives existence of a subsequence of $\{\Sigma_{t_j}\}$ converging to an integral p-current Σ on any compact set in the weak topology. The limit of a sequence of minimizing currents is minimizing, so Σ is absolutely area-minimizing. Because for all K we have

$$\partial(\Sigma) \llcorner K = \lim_{i \to \infty} \partial(\Sigma_{t_i}) \llcorner K$$

we have $\partial(\Sigma) \llcorner K = 0$ so Σ is a complete locally integral p-current. The asymptotic boundary of Σ lies in M since

$$\operatorname{supp}(\Sigma) \subset \operatorname{Conv}(M) \text{ and } \operatorname{Conv}(M) \cap \mathbb{S}^{n-1}(\infty) = M$$

Suppose $x \notin \overline{\operatorname{supp} \Sigma}$. Then there is a neighborhood N of x such that $N \cap \overline{\operatorname{supp} \Sigma} = \varnothing$. But if $x \in M$ we have $N \cap \operatorname{supp} \Sigma_t \neq \varnothing$ for sufficiently large t, a contradiction. Hence $M \subset \overline{\operatorname{supp} \Sigma} \implies M \subset \partial \Sigma$. That is, the asymptotic boundary of Σ is equal to M. This completes the proof. □

3.2 Properties of Minimal Hypersurfaces

When we further restrict the dimension, we get similar results for a more general class of asymptotic boundaries. The proof outline is analogous to that of the previous theorem.

3.2.1 Existence

Theorem 3.5. *Let $S \subset \mathbb{S}^{n-1}(\infty)$ be a closed set such that $\mathbb{S}^{n-1}(\infty) \setminus S$ has exactly two connected components. Suppose that there are $(n-2)$-dimensional smooth, closed, connected manifolds $M_j \subset \mathbb{S}^{n-1}(\infty)$ such that*

$$\lim_{j \to \infty} \rho(M_j, S) = 0$$

where ρ is the Hausdorff distance between sets. Then there exists an absolutely area-minimizing integral $(n-1)$-current Σ asymptotic to S at infinity.

Proof. Choose an origin $\mathcal{O} \in \mathbb{H}^n$ and let $M_j \subset \mathbb{S}^{n-1}(j)$ be a geodesic projection from $\mathcal{O}$. Let Σ_j be an integral $(n-1)$-current such that it is a solution to the Plateau problem with boundary M_j:

$$\partial\Sigma_j = M_j \qquad \underline{M}(\Sigma_j) \leq \underline{M}(\varphi)$$

for φ any integral $(n-1)$-current with $\partial\varphi = M_j$. As before, we want to show estimates

$$c_r \leq \underline{M}(\Sigma_j \llcorner B_r) \leq C_r$$

on the mass of Σ_j inside a geodesic r-ball B_r centered at $\mathcal{O}$. We first establish the following lemma.

Lemma 3.6. *Let Σ be an area-minimizing $(n-1)$ current in $B^n(s)$ with $\partial\Sigma = M$ a connected manifold in $\mathbb{S}^{n-1}(s)$. Then* $\operatorname{supp}\Sigma$ *is connected and disconnects $B^n(s)$ into two components $\Omega^\pm$*

Proof of Lemma 3.6. From Corollary 11.2 of [8], recall $\operatorname{supp}\Sigma = N \cup Z$ where N in analytic submanifold with $\partial N = \operatorname{supp}\partial\Sigma$ by connectedness of M and $Z \subset (\mathbb{R}^n \setminus N)$ is compact with Hausdroff dimension $\leq n-8$. This implies $Z \cap \operatorname{supp}\partial\Sigma = \varnothing$. Thus the boundary of each component of $\operatorname{supp}\Sigma$ is M, implying $\operatorname{supp}\Sigma$ is connected. For high co-dimension Z, we have $\pi_1(B^n(s) - \mathbb{Z}) = 0$.

Suppose $B^n(s) \setminus \operatorname{supp}\Sigma$ is connected. Choose a regular point $x \in \operatorname{supp}\Sigma$. That is, there is a neighborhood W of x such that $W \cap \operatorname{supp}(\Sigma)$ is a connected $(n-1)$-dim C^2-submanifold of $\mathbb{H}^n$. Choose L a transverse curve such that $L \cap \operatorname{supp}\Sigma = x$. Join the endpoints ∂L in $B^n(s) \setminus \operatorname{supp}\Sigma$ and obtain an embedding

$$f : \mathbb{S}^1 \to B^n(s)$$

such that $f(\mathbb{S}^1) \cap \operatorname{supp}\Sigma = x$. f extends to a map $f : D^2 \to B^n(s)$. Now assume that f is transverse to $\operatorname{supp}(\Sigma \setminus Z)$. Thus $f^{-1}(\operatorname{supp}(\Sigma \setminus Z))$ is a 1-manifold with single boundary component x, and so is homeomorphic to $\mathbb{R}_+$, a contradiction.

We now show there are at most *two* components of $B^n(s) \setminus \operatorname{supp}\Sigma$. Let x, L be as above and for any $y \in B^n(s) \setminus \operatorname{supp}\Sigma$ let τ_y be the shortest geodesic from y to $\operatorname{supp}\Sigma$. Let

$$\tau_y(0) = y \qquad \tau_y(1) = p$$

Then p is regular, see [4, 7.4.5]. We may join p and x by a path γ in the regular set of Σ. Now moving γ along L, in the direction normal to $\operatorname{supp}\Sigma$, we may construct a path in $B^n(s) \setminus \operatorname{supp}\Sigma$ from y to one endpoint of ∂L. Since L has two endpoints, there cannot exceed two connected components. □

Now we apply Lemma 3.6 to the current Σ_j in $B^n(j)$, to get that $\operatorname{supp}\Sigma_j$ separates $B^n(j)$ into two components. Letting $B^n(j) \setminus \operatorname{supp}\Sigma_j = \Omega_j^+ \cup \Omega_j^-$, we have $\Sigma_j = \partial\Omega_j^+$ and $\operatorname{vol}(\partial\Omega_j^+ \cap K) \leq \operatorname{vol}(\partial K \cap \Omega_j^+)$ for any compact $K \subset B^n(j)$. Choosing $K = B^n(r)$ for $r < j$, it follows that

$$\underline{M}(\Sigma_j \llcorner B_r) \leq \frac{1}{2} \operatorname{vol} \mathbb{S}^{n-1}(r)$$

for all j, since the equatorial r-disk has less area than either of the r-hemispheres. This gives the desired upper bound $C_r = \frac{1}{2}\operatorname{vol}\mathbb{S}^{n-1}(r)$.

For the lower bound, recall that given a set $T \subset \overline{\mathbb{H}^n}$ we may define a convex hull $\operatorname{Conv}(T)$. Recall useful results Lemma 3.4 and that for $T \subset \mathbb{S}^{n-1}(\infty)$, we have $\operatorname{Conv}(T) \cap \mathbb{S}^{n-1}(\infty) = \overline{T}$. Choose points x, y in different components of $\mathbb{S}^{n-1}(\infty) \setminus S$. Let γ be the unique geodesic asymptotic to x and y. For j sufficiently large, $\gamma \cap \mathbb{S}^{n-1}(j)$ consists of two points x_j, y_j with

$$x_j \to x$$
$$y_j \to y.$$

as $j \to \infty$ and x_j, y_j lie in distinct components of $\mathbb{S}^{n-1}(j) \setminus M_j$. By Lemma 3.6 $\operatorname{supp}\Sigma_j$ separates $B^n(j)$ into two components and so we have $\operatorname{supp}\Sigma_j$ intersects $\gamma \neq \varnothing$ for all j sufficiently large.

Since $\operatorname{supp}\Sigma \subset \operatorname{Conv}(M_j)$ for large enough j and $\operatorname{Conv}(M_j) \to \operatorname{Conv}(S)$ we see that the sequence

$$\{\operatorname{supp}\Sigma_j \cap \gamma\} \subset K$$

for some compact set K in the interior of $\mathbb{H}^n$. Thus there is a $p \in \gamma$ and $R > 0$ such that $\operatorname{dist}(p, \operatorname{supp}\Sigma_j) < R$ for all j. Therefore $\operatorname{supp}\Sigma_j$ intersects a fixed ball of radius R in $\mathbb{H}^n$, for each j. Theorem 3.1 ensures the existence of a lower bound c_r. So we have estimates

$$c_r \leq \underline{M}(\Sigma_j \llcorner B_r) \leq C_r$$

and the rest of the proof proceeds as in Theorem 3.2 □

3.2.2 Invariant Solutions

We now study such minimal hypersurfaces that are invariant under a discrete group of isometries acting on $\mathbb{H}^n$.

Let $O^+(n,1)$ be the group of orientation-preserving isometries of $\mathbb{H}^n$. Let Γ be a discrete subgroup of $O^+(n,1)$. Then the **limit set** Λ_Γ of Γ is the set of accumulation points of an orbit $\Gamma_x, x \in \mathbb{H}^n$, on $\mathbb{S}^{n-1}(\infty)$. Λ_Γ is a closed set which is minimal under the conformal automorphism of Γ on $\mathbb{S}^{n-1}(\infty)$. We have

$$\mathbb{S}^{n-1}(\infty) = \Omega_\Gamma \cup \Lambda_\Gamma$$

where Ω_Γ is the **domain of discontinuity** of Γ. Every point $x \in \Omega_\Gamma$ has a neighborhood U such that $U \cap g(U)$ is nonempty for only finitely many $g \in \Gamma$. Further, Ω_Γ may have one, two, or infinitely many components, or may be empty. Γ is **quasi-Fuchsian** if Ω_γ has exactly two components.

If Γ acts freely $\iff$ Γ is torsion free, then Γ is quasi-Fuchsian if and only if the quotient manifold (where $\mathscr{C}(\Lambda_\Gamma)$ denotes the **convex core**)

$$\mathscr{C}^n = \frac{\mathscr{C}(\Lambda_\Gamma)}{\Gamma} \subset \frac{\mathbb{H}^n \cup \Omega_\Gamma}{\Gamma} = M^n$$

is a convex hyperbolic manifold. That is, any path in $\mathscr{C}^n$ is homotoptic to a geodesic. Furthermore, there must be two boundary components strictly contained in $\mathring{M}^n$. Also note that

$$\pi_1(M^n) \simeq \pi_1(\partial M)$$

The main result is the following

Theorem 3.7. *Let Γ be a quasi-Fuchsian group acting on $\mathbb{H}^n$. Then there exists complete Γ-invariant absolutely area minimizing $(n-1)$-currents Σ_Γ in $\mathbb{H}^n$.*

Proof. Let M_i be a sequence of smooth manifolds in the interior of $\mathscr{C}(\Lambda_\Gamma)$ eventually lying outside any compact set in $\mathbb{H}^n$. By our assumption,

$$\mathbb{S}^{n-1}(\infty) \setminus \Lambda_\Gamma = \Omega_\Lambda$$

has two components so we may apply Theorem 3.5. Let Σ be a complete area-minimizing hypersurface in $\mathbb{H}^n$ asymptotic to $\Lambda_\Gamma \subset \mathbb{S}^{n-1}(\infty)$. We may assume $\operatorname{supp}\Sigma$ is connected. Then by Lemma 3.6, $\mathbb{H}^n \setminus \operatorname{supp}\Sigma$ has two components $\Omega^\pm$ such that $\Omega^\pm \cap \mathbb{S}^{n-1}(\infty)$ are two (Γ-invariant) components of $\mathbb{S}^{n-1}(\infty) \setminus \Lambda_\Gamma$. Now consider the currents $g\Sigma$ defined by

$$(g\Sigma)(w) = \Sigma(g^* w)$$

for $g \in \Gamma$, each a minimizing integral $(n-1)$-current. Furthermore it is a boundary of least area in the sense

$$\partial(g\Omega^\pm) = g\Sigma$$

where $g\Omega^\pm$ are the components of $\mathbb{H}^n \setminus \operatorname{supp}(g\Sigma)$. Now consider

$$\Omega_1 = \bigcap_{g\in\Gamma} g\Omega^+$$

which is Γ-invariant, and hence so is $\partial\Omega_1$. If $\partial\Omega_1$ is a boundary of least area, we are done. But if not, we solve the Plateau problem in Ω_1 in the following way:

Let B_i be a sequence of smooth connected $n-2$ manifolds in $\Omega_1 \cap \mathscr{C}(\Lambda_\Gamma)$ eventually asymptotic to the sphere at infinity. Let φ_i be a solution of the Plateau problem with boundary B_i. We claim that $\varphi_i \subset \Omega_1$ for all i. One has $B_i \subset \Omega_1$ so that in particular it's a subset of $g\Omega^+$ for any $g \in \Gamma$. Since $g\Omega^+$ has boundary of least area, $\varphi_i \subset g\Omega^+$ for any g thus proving the claim. Now there is a sequence of boundaries of least area $\{\varphi_i\}$ in Ω_1 with $\{\partial\varphi_i\}$ converging to Λ_Γ in the Hausdorff distance. Now applying the proof of Theorem 3.5 to extract a convergent subsequence, again called $\{\varphi_i\}$, such that $\varphi_i \rightharpoonup \varphi^1$ weakly, with $\operatorname{supp}\varphi^1 \subset \Omega_1$. Now φ^1 is a boundary of least area. We may define an ordering $<$ on the set of complete minimal currents asymptotic to Λ_Γ by

$$\Sigma_1 < \Sigma_2 \iff \Omega_1^+ \supset \Omega_2^+$$

where $\Omega_i^+ \cap \mathbb{S}^{n-1}(\infty)$ is the positive component of $\mathbb{S}^{n-1}(\infty) \setminus \Lambda_\Gamma$. Hence $g\Sigma < \varphi^1$ for all $g \in \Lambda$. We can repeat the process on φ^1. If φ_1 is not Λ-invariant, let

$$\Omega_2 = \bigcap_{g \in \Lambda} g(\Omega_1)^+$$

where $g(\Omega_1)^+$ is the positive component of $\mathbb{S}^{n-1}(\infty) \setminus \Lambda_\Gamma$. Continuing, we produce a sequence of boundaries of least area φ^i such that

$$\Sigma = \varphi^0 < \varphi^1 < \cdots < \varphi^k < \ldots$$

and also $g\varphi^i < \varphi^{i+1}$ for all $g \in \Gamma$ and for all i. Each φ_i is a complete area minimizing $(n-1)$-current asymptotic to Λ_Γ and satisfying $\operatorname{supp} \varphi^i \subset \mathscr{C}(\Lambda_\Gamma)$. Applying the extraction argument again, we get a convergent subsequence $\varphi^{k'}$ that converges to Σ_Γ weakly, and it's clear that the limit is a complete area-minimizing integral $(n-1)$-current asymptotic to Λ_Γ.

Note that

$$\Sigma_\Gamma = \lim_{k \to \infty} \varphi^k$$

so

$$g\Sigma_\Lambda = \lim_{k \to \infty} g\varphi^k$$

and by construction $g\varphi^k < \varphi^{k+1}$, implying $g\Sigma_\Gamma \leq \Sigma_\gamma$. For any $g \in \Gamma$, $g^{-1} \leq \Sigma_\Gamma$ also implies $\Sigma_\Gamma \leq g\Sigma_\Gamma$, this gives us $g\Sigma_\Gamma = \Sigma_\Gamma$ for all $g \in \Gamma$. □

3.3 Minimal surfaces in hyperbolic 3-manifolds

We will cite some results of Anderson, but refer to [2] for the proofs. The results are not referenced to any further, just interesting to note. We then proceed to some results by Alexakis-Mazzeo [3], where they study the renormalized area of minimal surfaces in the same setting.

Theorem. *Let γ be a Jordan curve on $\mathbb{S}^2(\infty)$. Then there exists a complete embedded minimal surface D in $\mathbb{H}^3$, homeomorphic to a disk, asymptotic to γ. Furthermore, D minimizes area in the category of embedded disks.*

Sketch. Theorem 4.1 in [2]. In summary, we want to apply the proof of Theorem 3.5 to a sequence of smoothly embedded minimal disks obtained by results of Almgren-Simon. To establish necessary estimates we can use the monotonicity formula. Then it is left to show the limiting surface is a smooth completely embedded disk. □

Corollary. *Let Γ be a quasi-Fuchsian group acting on $\mathbb{H}^3$. Then there is a Γ-invariant complete smoothly embedded minimal disk $\widetilde{D}$ in $\mathbb{H}^3$.*

Sketch. Uses the previous theorem and an argument similar to Theorem 3.7 □

Proposition. *There exists Jordan curves γ on $\mathbb{S}^2(\infty)$ such that any absolutely area minimzing surface Σ asymptotic to γ has genus $g > g_0$, for any prescribed $g_0 \geq 0$.*

Theorem. *There exists torsion-free quasi-Fuchsian groups Γ_g such that any complete area-minimizing Γ_g-invariant surface in $\mathbb{H}^3$ has infinite genus.*

Now moving on to [3]. Note that there is a well-defined Hadamard regularization of the area of minimal surfaces in hyperbolic 3-manifolds, described as follows. We work in the Poincaré half-space model with coordinates (y, x) for $y \in \mathbb{R}^2, x > 0$. For any γ at $\mathbb{S}^2(\infty)$ we can find the appropriate parametrization for γ in $\mathbb{R}^2$. Let $\Gamma = \{(y, x) \in \mathbb{R}^2 \times \mathbb{R}^+ : y \in \gamma\}$ denote the **vertical cylinder** over γ. Choose a family of minimal hemispheres osculating γ at each point and lying completely on either side of Γ respectively. The $\Gamma^{\pm}$ envelopes of these families are smooth convex surfaces and act as barriers for any minimal surface asymptotic to γ by the maximum principle.

Consider a minimal surface Y in $\mathbb{H}^3$ asymptotic to a curve γ at asymptotic infinity. By the above we can also conclude that Y intersects $\mathbb{R}^2$ orthogonally along γ. We may also write Y as a normal graph over Γ in the obvious way. Letting $N(s, x)$ be the unit normal of Γ, there is some scalar function $u(s, x)$ such that

$$Y = \{(\gamma(s) + N(s, x)u(s, x), x)\}$$

in a sufficiently small neighborhood of a point of γ. The barrier argument then implies $u(s, 0) = \partial su(s, 0) = 0$.

Writing $Y_\varepsilon = Y \cap \{x \geq \varepsilon\}$, the **renormalized area** is the constant term in the expansion

$$\int_{Y_\varepsilon} dA = \frac{\text{length}(\gamma_\varepsilon)}{\varepsilon} + \mathcal{A}(Y) + \mathcal{O}(\varepsilon)$$

This $\mathcal{A}(Y)$ is well-defined, a result by Graham and Witten. Alexakis and Mazzeo [3] study this renormalized area functional along with its natural domain, the moduli space of all properly embedded minimal surfaces with embedded asymptotic boundary. The first result of which is an explicit formula for the renormalized area

Theorem 3.8. *Let (M, g) be a hyperbolic manifold (more generally a Poincaré-Einstein space) and $\gamma \subset \partial M$ a $C^{3,\alpha}$ embedded closed curve, and suppose that $Y^2 \subset M$ is a properly embedded minimal surface with asymptotic boundary γ. Then the renormalized area of Y is given by*

$$\mathcal{A}(Y) = -2\pi\chi(Y) - \frac{1}{2}\int_Y \left|\hat{k}\right|^2 dA + \int_Y W_{1212} dA$$

where $\hat{k}$ is the trace-free second fundamental form of Y and W_{1212} is the Weyl curvature of g.

Proof. Denote $R_{ijkl}, (R_Y)_{ijkl}$ components of the curvature tensor of g and $g|_Y$. Einstein property means that the Ricci curvature satisfies $R_{ij} = -ng_{ij}$. From the decomposition of the curvature of Einstein metrics, we have that the Weyl tensor for g is

$$W_{ijkl} = R_{ijkl} + g_{ik}g_{jl} - g_{il}g_{jk}$$

Now pick $p \in Y$ and an oriented orthononormal basis $\{e^1, \dots, e^{n+1}\}$ for T_pM such that $\{e^1, e^2\}$ is an oriented basis for T_pY. Denote

$$k^s_{ij}, i, j = 1, 2 \qquad s = 3, \dots, n+1$$

the components of the second fundamental form of Y at p. The Gauss-Codazzi equations then read

$$R_{1212} = (R_Y)_{1212} + \sum_{s=3}^{n+1} \left(k^s_{11}k^2_{22} - k^s_{12}k^s_{12}\right)$$

For each s, $k^s_{11} + k^s_{22} = (\hat{k}^s_{11} + H^s) + (\hat{k}^s_{22} + H^s) = 2H^s$ and hence

$$\begin{aligned}
\sum_{s=3}^{n+1} \left(k^s_{11}k^2_{22} - k^s_{12}k^s_{12}\right) &= \sum_{s=3}^{n+1} \left(\hat{k}^s_{11} + H^s\right)\left(\hat{k}^s_{22} + H^s\right) - \left(\hat{k}^s_{12}\right)^2 \\
&= \sum_{s=3}^{n+1} \hat{k}^s_{11}\hat{k}^s_{22} + \hat{k}^s_{11}H^s + \hat{k}^s_{22}H^s + (H^s)^2 - \left(\hat{k}^s_{12}\right)^2 \\
&= \sum_{s=3}^{n+1} \hat{k}^s_{11}\hat{k}^s_{22} + (H^s)^2 - \left(\hat{k}^s_{12}\right)^2 \\
&= \sum_{s=3}^{n+1} (H^s)^2 - \frac{1}{2}\left[\left(\hat{k}^s_{11}\right)^2 + \left(\hat{k}^s_{22}\right)^2 - 2\left(\hat{k}^s_{12}\right)^2\right] \\
&= |H|^2 - \frac{1}{2}\left|\hat{k}\right|^2
\end{aligned}$$

So R_{ijkl} becomes

$$R_{1212} = (R_Y)_{1212} - |H|^2 + \frac{1}{2}\left|\hat{k}^2\right|$$

Now plug in $W_{1212} = g_{11}g_{22} - g_{12}g_{12} + R_{1212}$ to get

$$(R_Y)_{1212} + \frac{1}{2}\left|\hat{k}\right|^2 - |H|^2 - W_{1212} = g_{12}g_{12} - g_{11}g_{22} = -1$$

where $K = (R_Y)_{1212}$ is the Gauss curvature of Y. Integrating everything over $Y_\varepsilon = Y \cap \{x \geq \varepsilon\}$, we obtain

$$\int_{Y_\varepsilon} K dA - \frac{1}{2}\int_{Y_\varepsilon} \left(2|H|^2 - \left|\hat{k}^2\right|\right) dA - \int_{Y_\varepsilon} W_{1212} dA = -\int_{Y_\varepsilon} dA$$

the regularity of our boundary curves implies that for ε small enough, we have $\chi(Y_\varepsilon) = \chi(Y)$, so by the Gauss-Bonnet theorem we have

$$\int_{Y_\varepsilon} K dA = 2\pi\chi(Y) - \int_{\gamma_\varepsilon} \kappa_\varepsilon ds$$

where κ_ε is the geodesic curvature of the boundary $\gamma_\varepsilon := \partial Y_\varepsilon$ and ds the length element with respect to the induced metric $g|_{\gamma_\varepsilon}$. We then have

$$\int_{Y_\varepsilon} dA = -2\pi\chi(Y) + \int_{\gamma_\varepsilon} \kappa_\varepsilon ds + \frac{1}{2}\int_{Y_\varepsilon} \left(2|H|^2 - \left|\hat{k}^2\right|\right) dA + \int_{Y_\varepsilon} W_{1212} dA$$

Now as $\varepsilon \to 0$, we use the formula

$$\int_{\gamma_\varepsilon} \kappa_\varepsilon ds = \frac{\text{length}(\gamma_\varepsilon)}{\varepsilon} + \mathcal{O}(\varepsilon) \tag{1}$$

we obtain

$$\int_{Y_\varepsilon} dA = -2\pi\chi(Y) + \mathcal{A}(Y) + \frac{\text{length}(\gamma_\varepsilon)}{\varepsilon} + \mathcal{O}(\varepsilon)$$

with the renormalized area given by

$$\mathcal{A}(Y) = -2\pi\chi(Y) + \lim_{\varepsilon\to 0}\left(\frac{1}{2}\int_{Y_\varepsilon}\left(2|H|^2 - \left|\hat{k}^2\right|\right) dA + \int_{Y_\varepsilon} W_{1212} dA\right)$$

Now recall the transformation law $\hat{k}_{ij}(e^{2\phi}g) = e^{\phi}\hat{k}_{ij}(g)$ for the trace free second fundamental form under a conformal change of the ambient metric $g \to e^{2\phi}g$. When Y is 2 dimensional,

$$\left|\hat{k}(e^{2\phi}g)\right|^2_{e^{2\phi}g} dA_{e^{2\phi}g} = \left|\hat{k}(g)\right|^2_g dA_g$$

and the components of the Weyl tensor transform as

$$W_{1212}(e^{2\phi}g) dA_{e^{2\phi}g} = W_{1212}(g) dA_g$$

thus the second and third terms have a limit. □

Foreshadowing to the next section (terms will be defined there), we have the following remark.

Remark 3.1. The **entanglement entropy** of a domain $A \subset \mathbb{S}^2$ corresponds to the *renormalized area* of the minimal surface $Y \subset \mathbb{H}^3$ with boundary $\gamma = \partial A$.

The next result of [3] concern the natural domain of $\mathcal{A}$, namely the moduli space of all properly embedded minimal surfaces with embedded asymptotic boundary.

Let (M, g) be a convex cocompact hyperbolic 3-manifold and fix $k \in \mathbb{Z}_{\geq 0}$. Now define $\widetilde{\mathcal{M}}_k(M)$ to be the space of all properly embedded surfaces of genus k which extend to $\overline{M}$ as $\mathcal{C}^{3,\alpha}$ submanifolds with boundary intersecting ∂M orthogonally. Define $\mathcal{M}_k(M) \subset \widetilde{\mathcal{M}}_k(M)$ the subspace of all such surfaces that are minimal surfaces. Also denote $\mathcal{E}$ the space of all $C^{3,\alpha}$ closed embedded curves $\gamma \subset \partial M$. Both $\widetilde{\mathcal{M}}_k(M)$ and $\mathcal{E}$ are Banach manifolds, see for example [9].

Proposition 3.9. *For each k, $\mathcal{M}_k(M)$ is a Banach manifold.*

Proof Sketch. Fix any $Y \in \mathcal{M}_k(M)$ and assume that $\partial Y = \gamma$ is a C^∞ embedded curve in ∂M. The goal is to construct a coordinate chart containing Y in $\mathcal{M}_k(M)$ which maps, in the *non-degenerate* setting, to a small ball around 0 in the space of Jacobi fields which are $C^{3,\alpha}$ up to ∂Y. Further, the ball is identified with a small ball in the space of $C^{3,\alpha}$ normal vector fields along γ.

Let ν be a unit normal vector field along Y and ϕ any sufficiently small scalar $C^{3,\alpha}$ function on Y. We can define the normal graph over Y by

$$Y_{0,\phi} = \{\exp_p(\phi(p)\nu(p) : p \in Y)\}$$

which is equivalently a small perturbation of Y. The mean curvature of $Y_{0,\phi}$ is a nonlinear elliptic second order operator $\mathcal{F}(\phi)$ with linearization

$$D\mathcal{F}|_{\phi=0} := L_Y = \Delta_Y + |A_Y|^2 - 2$$

where A_Y denotes the second fundamental form of Y. The Jacobi operator L_Y is elliptic uniformly degenerate operator of order 2.

It is necessary to consider broadly deformations of Y where γ also varies. Let $\overline{\nu} = x^{-1}\nu$ be the unit normal to Y under the conformally compactified metric $\overline{g} = x^2 g$. We know $\overline{\nu}$ extends smoothly to $\overline{Y}$ and its restriction to γ is the unit normal $\overline{N}$ to this curve in ∂M with respect to the boundary metric h_0. Any nearby curve can then be written

$$\gamma_\psi = \{\exp_p(\psi(p)\overline{N}(p)) : p \in \gamma\}$$

Now define an extension operator $\mathcal{E}$ which associates a surface $Y_{\psi,0}$ which is 'approximately minimal' and which has $\partial Y_{\psi,0} = \gamma_\psi$: Letting u be the graph function of Y over the vertical cylinder Γ, let u_ψ be a new graph function in some ε-neighborhood of the boundary such that $u_\psi(s,0) = \psi(s)$ and $\partial_x^j u_\psi(s,0)$ depends on formal expansions of solutions for $\mathcal{F}$ for $j = 1, 2$. Also set the $j = 3$ derivative to $u_3(s)$, the Cauchy data for Y. Now let

$$U_\psi = \chi u_\psi + (1 - \chi)u$$

where χ is a cutoff function which equals 1 near $x = 0$. It can then be checked that $\mathcal{F}(U_\psi) \in x^\mu C^{1,\alpha}(Y)$ for $0 < \mu < 2$. We then have that

$$w := D\mathcal{E}|_0(\hat{\psi}) \sim x^{-1}\hat{\psi}$$

as $x \to 0$ and $L_Y w = \mathcal{O}(x^\mu)$ for some $\mu \in (0, 2)$. Finally write a new perturbed surface over $Y_{\psi,0}$ with a certain graph function ϕ that decays to order μ, denoted $Y_{\psi,\phi}$. It's mean curvature is written $\mathcal{F}(\psi, \phi)$. A neighborhood of Y in $\mathcal{M}_k(M)$ is identified with the space of solutions to $\mathcal{F}(\psi, \phi) = 0$. Using the implicit function theorem and treating the degenerate and non-degenerate cases separately, it can be shown that $\mathcal{F}(\psi, \phi)$ is always surjective. Along with the observation that $\mathcal{F} : \mathcal{B} \to x^\mu C^{1,\alpha}(Y)$ is a smooth mapping when $\mathcal{B}$ is a small origin neighborhood in $C^{3,\alpha}(\gamma) \times x^\mu C^{3,\alpha}(Y)$, we get that $\mathcal{M}_k(M)$ is a smooth Banach manifold in a neighborhood of Y. $\square$

The final results of interest concern the mapping between a hyperbolic minimal hypersurface and its boundary at infinity.

Proposition 3.10. *The natural map*

$$\Pi : \mathcal{M}_k(M) \to \mathcal{E}(\partial M)$$

assigning to any such $Y \in \mathcal{M}_k(M)$ its asymptotic boundary $\partial Y = \gamma$ is Fredholm with index 0*.*

Proposition 3.11. Π *is a proper mapping.*

Taken together, these results provide a sort of analogue to the Manifold Structure Theorem of [10] in our asymptotic boundary case.

4 Holographic Entanglement Entropy

What follows is an informal, nonrigorous discussion of related physical ideas in quantum gravity. We outline some aspects of the black hole information paradox and proposed resolutions, and we present physical problems which raise interesting geometric questions. In particular, we will outline a specific paper [11] that physically motivates some mathematical work on mountain pass theorems in (asymptotically) hyperbolic space.

4.1 Background

Recall that the theory of general relativity predicts the existence of black holes, regions of spacetime such that nothing can escape. Stephen Hawking considered the formalism of quantum mechanics in such systems and discovered that an isolated black hole admits a form of "Hawking" radiation. Hawking furthermore argued that the radiation would be independent of the initial state of the black hole and would depend only on its mass, electric charge, and angular momentum.

The **black hole information paradox** is as follows: consider a scenario in which a black hole is formed and then evaporates entirely through Hawking radiation. Hawking showed that final state of radiation retains information only about the total mass, electric charge, and angular momentum of the initial state. These quantities do not uniquely define a physical state, hence this suggests that many initial physical states could evolve into the same final state. Therefore, information about the details of the initial state would be permanently lost. However, this violates a principle of both classical and quantum physics. In quantum mechanics specifically, the state of the system is encoded by its wave function. The evolution of the wave function is determined by a unitary operator, and unitarity implies that the wave function at any time can determine the wave function of any other time in the past or future.

Recall that an anti de-Sitter space is a maximally symmetric Lorentzian manifold with constant negative scalar curvature, and so its boundary is "infinitely far away". Regarding the potential resolution to the information paradox, an important idea is the **anti de-Sitter/conformal field theory (AdS/CFT) correspondence**, which states that the boundary of anti-de Sitter space can be regarded as the "spacetime" for a conformal field theory. The claim is that this conformal field theory is equivalently correspondent to the gravitational theory on the bulk anti-de Sitter space.

4.2 Holographic Entanglement Entropy

Progress in this area in the past two decades has suggested that information does, in fact, escape black holes via their radiation. We outline the developments of the geometric objects arising below. The most recent of which (EW surfaces) seems to encode the amount of information that has radiated away from the black hole. These surfaces evolve over the black hole's lifetime precisely as expected if information escapes.

4.2.1 Classical case

Physicists Ryu and Takayanagi [12] conjecture that, given some static AdS spacetime M with subregion $A \subset \partial M$ in the asymptotic boundary, a CFT on A has Von Neumann entanglement entropy given by $S \propto \text{Area}(Y_A)$, where Y_A is a minimal surface in M with $\partial Y_A = A$. Intuitively, view S_A as the entropy for a physical observer who is only accessible to the subsystem A and cannot receive any signals from $B = \partial M \setminus A$. In this sense, the subsystem B is analogous to the inside of a black hole horizon for an observer sitting in A, i.e., outside of the horizon.

Let us be more precise: in Lorentzian spacetimes, we cannot in general define a minimal surface because by variations of a spacelike surface in the time direction, we may make its area arbitrarily small. For static spacetimes, we can circumvent this problem by Wick rotating to Euclidean signature $\iff$ restricting attention to a constant time slice. The spacelike foliation of ∂M extends into the bulk to provide a spacelike foliation $\bigcup_t N_t$ of M. On a given spacelike slice in M, construct a minimal surface Γ_A anchored at $\partial A \subset \partial N$.

Ryu-Takayanagi propose that the entanglement entropy of a conformal field theory on subregion A is given by area of a minimal bulk surface Γ_A

$$S_{vN}[A] = \min_{A \sim \Gamma_A} \frac{\text{area}[\Gamma_A]}{4G\hbar}$$

where A is homologous to Γ_A. Restricting to spatial slice of anti de-Sitter space, from the above we have a minimal surface with ideal boundary at asymptotic infinity. For M some (asymptotically) hyperbolic manifold, [3] expresses the renormalized area of a properly embedded minimal surface Y in M in terms of its Euler characteristic and an integral of local invariants, and it is this area we are interested in for the holographically associated system at infinity. The above constructions are often denoted as **RT Surfaces**.

For general spacetimes, Hubeny, Rangamani, and Takayanagi [13] seek a covariant generalization to the above minimal-surface prescription, now known as

HRT surfaces. Consider a first prescription. For asymptotically AdS spacetimes one has a foliation by zero mean curvature surfaces i.e maximal area spacelike slice through the bulk anchored on ∂N. Denote the leaves of maximal-area foliation by Σ_t. Choose a slice, and on this slice we construct a minimal-area surface anchored on ∂A_t, denoted X.

$$S_A = \frac{\text{Area}(X)}{4G\hbar}$$

Now consider second prescription. Let W be an extremal surface homologous to A, given by an extremal of the area action. We then follow a generalization of the RT surface argument to Lorentzian signature.

Claim. *X coincides with W if X is constructed on totally geodesic spacelike surfaces.*

Definition 4.1. Among all *extremal* surfaces of the area functional, choose the minimal area surface. This is a **HRT Surface**.

Wall [14] argues certain results of quantum information theory, namely monogamy of mutual information, strong subadditivity, entanglement wedge nesting. Wall also argues the equivalence of HRT surface and classical maximin surface. Maximin surfaces are defined by minimizing area on some slice and then maximizing the area with respect to varying slices. As an important remark, any extremal surface σ has a representative $\tilde{\sigma}$ on any other time slice Σ, defined by sending out null congruences C and defining $\tilde{\sigma} = C \cap \Sigma$. If σ is an extremal surface, we have $\text{Area}(\sigma) \geq \text{Area}(\sigma)$ by Raychaudhuri equation. This allows one to project all extremal surfaces onto the same slice Σ, enabling arguments of dynamical cases to be reduced to arguments of the static case.

4.2.2 Quantum case

In [15], Engelhardt and Wall proposed an extension of the classical case to account for quantum effects. Their prescription is that the entanglement entropy of a region A is given by the generalized entropy of the minimal quantum extremal surface Γ_A

$$S_{vN}[A] = \underset{A\sim\Gamma_A}{\text{ext}}\, S_{gen}[\Gamma_A]$$

$$S_{gen}[\Gamma_A] = \left(\frac{\text{Area}(\Gamma_A)}{4G} + \dots\right) - \text{Tr}\rho\log\rho$$

Where ρ the density matrix of matter fields outside the black hole which geometrically is the region Γ_A "encloses". Here a quantum extremal surface is a stationary point of the total generalized entropy S_{gen}. Furthermore, we have the following:

Definition 4.2. A **quantum maximin surface** is obtained by the following procedure: For every time slice containing A, find the minimal S_{gen} surface homologous to A. Find the maximum S_{gen} surface among all these minima.

Claim. *A quantum maximin surface is identical to the* ***EW surface*** *i.e. minimal quantum extremal surface [Akers et al.]*

It is evident that the prescription here is analogous to the notion of a h-prescribed mean curvature (PMC) surface $\Sigma = \partial\Omega$, that is a critical points of the functional

$$\mathcal{A}^h = \text{Area} - \int_\Omega h$$

where the added term is the enclosed h-volume. In [16] Zhou and Zhu develop a PMC min-max theory. They show that for generic set of smooth prescription functions h on a closed ambient manifold, there always exists a nontrivial, smooth, closed hypersurface of prescribed mean curvature h. In order to make progress towards a rigorous notion to Engelhardt and Wall's quantum extremal surfaces, an open problem is to develop a PMC min-max theory on (asymptotically) hyperbolic or conformally compact manifolds.

4.3 Python's Lunch

We now discuss a recent result [11] relating the above setup to a mountain-pass type problem in geometry; it is known as a "Python's Lunch".

In physics, there is a useful model of black holes as quantum computers in the following sense: Take a collection of N qubits which evolve under the action of a k-local Hamiltonian H. Here H is a Hermitian matrix which can be represented as the sum of m Hamiltonian terms, acting upon at most k qubits each. Now enforce that the number of qubits is determined by the entropy of the black hole, $N \sim S_{BH}$. Using the quantum computing language, we may talk about the computational complexity of processes. In particular for any unitary operator U, one definition of the complexity of U is that it is the smallest number of 2-qubit quantum logic gates g needed to represent it. As in for an operator U of complexity n, we have

$$U = g_1 \dots g_n$$

In [17], Harlow and Hayden studied the complexity of obtaining a single qubit of information from Hawking radiation, and they argue that the complexity of recovering information grows exponentially with the entropy S_{BH} of the black hole. The Python's Lunch paper connects the argument by Harlow and Hayden to a problem in geometry.

We now outline their procedure for decoding the information while acting solely on the Hawking radiation. By ER=EPR [18] it is known that if we consider a black hole, and somehow capture the radiation of the original black hole and collapse it into the second black hole, then the two black holes are entangled. The entanglement can then be interpreted as a wormhole[1] (Einstein-Rosen Bridge) connecting the two black holes. Now, at 'Page time' the wormhole would have volume proportional to S_{BH}^2 which they state is *far less than the exponential complexity claimed by Harlow and Hayden, but still too large to*

[1]Refer to a mathematically rigorous definition of a wormhole by Wong.

easily implement the decoding. The next step now is to *apply unitary operations to the second black hole* to *shorten the wormhole* and bring it to the Thermofield Double State. In this state there is no separation between the two horizons of the black holes, and decoding the information would be easy.

Now the only difficult step in the outlined procedure is shortening the wormhole, and since Harlow and Hayden argue that decoding information from the radiation alone is exponentially hard, the authors conclude that shortening the wormhole must be exponentially hard. The argument of the Python's Lunch paper is that there is a geometric obstruction causing this exponential difficulty which is defined below. The physical idea is that there is a bulge in the wormhole, and "passing through" such bulge takes exponential complexity.

The python's lunch geometry is a wormhole with a bulge. As in there are three regions of length L_i, $1 \leq i \leq 3$ depending polynomialy on N, where the inner region has larger area than the outer region. An alternative way to look at this geometry is as the tensor network (TN) which prepares a two-sided quantum state. Recall that the QM wavefunction can be represented as a tensor contraction of a network of individual tensors. A tensor network is a graphical way of portraying this information. Each leg of a tensor portrays tensor contraction in a precise sense. In AdS/CFT the area ($A/4G_N$) of the cut corresponds to the number of tensor network legs. It is a fact about tensor networks that entropy of a system is upper bounded by the minimal cut through the network.

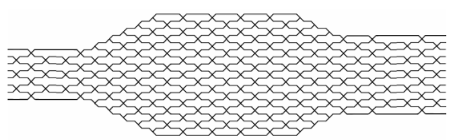


Figure 6: The tensor network that corresponds to the Python's Lunch geometry. The throats and bulge (where the girth is constant) are composed of unitary gates, whereas the shoulders (where the girth changes) involve projections like those shown in Fig. 7.

The authors finally define

Definition 4.3. Python's Lunch geometry is a particular set of three extremal surfaces. They are the two end surfaces Σ_1, Σ_2 and the bulge surface Σ in the middle, as follows: Σ_1 is the minimal area HRT surface homologous to one end of the wormhole. It is analogous to the minimal cut through a tensor network. Σ_2 a second minimal surface, homologous to the first. It is analogous to a second locally minimal cut through a tensor network. The **bulge surface** Σ is a third minimal surface that lies in between the first two surfaces, and has a larger area than both, and is homologous to both.

For spacetimes where quantum effects are important, such as evaporating black holes, it is more accurate to consider Engelhardt-Wall surfaces in place of HRT surfaces. Nevertheless, the authors then have the following conjecture.

Conjecture 4.1. *In a covariant Python's lunch geometry with min-max-min entropies S_L^{gen}, S_{max}^{gen}, S_R^{gen}, and with the assumption $S_L^{gen} < S_R^{gen}$ the restricted complexity of the system is*

$$\mathcal{C}_R[U_{PL}] = (const) \times \mathcal{C}_{TN} \cdot \exp\left[\frac{1}{2}\left(S_{max}^{gen} - S_R^{gen}\right)\right]$$

where $\mathcal{C}_{TN}$ denotes the size of the tensor network.

Every example that the authors consider of a spacetime with more than one extremal surface sharing the same boundary will turn out to have a Python's Lunch. Furthermore [11, Section 5] demonstrates why the Python's lunch geometry explains the exponential complexity in decoding Hawking Radiation. This motivates a mountain pass result in hyperbolic spaces in the next section, and more generally in conformally compact manifolds.

5 Mountain Pass in Hyperbolic Space

During our work there was a preprint [19] released that addresses this by developing a min-max theory in hyperbolic space. The argument in this section is different in that it uses a result by [20] applied to each domain in a compact exhaustion to obtain a local approximation. The result we use from [20], based on the work of [21], is the following:

Theorem. *Let M^{n+1} be a compact, oriented, Riemannian manifold with strictly convex boundary, and γ^{n-1} be a closed, embedded, oriented, smooth submanifold of ∂M. Suppose that there exist distinct embedded, oriented, smooth, strictly stable minimal hypersurfaces Γ_1 and Γ_2 , such that $\partial\Gamma_i = \gamma$, for $i = 1, 2$, and Γ_1 and Γ_2 are homologous. Suppose also that all connected components of each Γ_i have non-empty boundary. Then there exists a distinct embedded minimal hypersurface Σ in M with $\partial\Sigma = \gamma$ and there is a connected component of Σ which is contained neither in Γ_1, nor in Γ_2 .*

Now let $\mathbb{H}^3$ be hyperbolic space along with its standard compactification giving the boundary at infinity. We work in the Poincaré ball model and upper half-space model.

Theorem 5.1 (Main Theorem)**.** *Let γ be a closed, embedded, oriented, smooth submanifold of $\partial\mathbb{H}^3 = \mathbb{S}^2(\infty)$. Suppose that Σ_1 and Σ_2 in $\mathbb{H}^3$ are two distinct embedded, oriented, smooth, strictly stable minimal hypersurfaces such that $\partial\Sigma_i = \gamma$, for $i = 1, 2$. Then there exists a distinct embedded minimal surface Σ in $\mathbb{H}^3$ with $\partial\Sigma = \gamma$.*

Setup

$n = 3$. Choose a point $0 \in \mathrm{Conv}(\gamma)$ in $\mathbb{H}^n$ and retract γ smoothly to 0 via a geodesic flow. Let

$$\gamma_t = \{p : p \in \eta_q(t) \mid \eta_q \text{ is a geodesic from 0 to } \gamma\}$$

so that γ_t is a smooth 1-parameter family of embedded manifolds each in a geodesic t-sphere $\mathbb{S}^{n-1}(t)$ centered at 0. In particular we consider geodesic balls $\mathbb{B}^n(t)$ with boundary $\mathbb{S}^{n-1}(t)$ as a compact exhaustion of $\mathbb{H}^n$. Then on each compact region and for sufficiently large t, we have $\Sigma_{1,t} = \Sigma_1 \cap \mathbb{S}^{n-1}(t)$ and $\Sigma_{2,t} \cap \mathbb{S}^{n-1}(t)$ with $\partial\Sigma_{i,t} = \gamma_t$ for $i = 1, 2$. Each $\Sigma_{i,t}$ is thus converging to Σ_1 and Σ_2 respectively.

From the main result in [20] by Montezuma, we have that on each compact domain $\mathbb{B}^n(t)$ there exist a third distinct embedded hypersurface $\Sigma_{3,t} := \Sigma_t$ with $\partial\Sigma_t = \gamma_t$. Applying the standard existence argument from Anderson [1], we get that Σ_t converges to some $\Sigma_3 = \Sigma$ with $\partial\Sigma = \gamma$.

it remains to make sure that Σ is distinct from $\Sigma_i, i = 1, 2$. We first need to establish a few results. Let x be a boundary defining function for $\partial\mathbb{H}^n$. In particular, we work with the upper half space model for hyperbolic space.

Lemma 5.2. *Let $\widetilde{Y}$ be minimal surface in $\mathbb{H}^3$ with $\gamma = \partial\widetilde{Y} \subset \partial\mathbb{H}^3$, and suppose $\widetilde{Y}$ is intersecting $\partial\mathbb{H}^3$ orthogonally. Suppose we have a minimal surface Y written as a normal graph over $\widetilde{Y}$ via the function $u \in C^\infty(\widetilde{Y})$. If $\partial Y = \gamma$ then u vanishes to order x^2 in a neighborhood of ∂Y.*

Proof. In the upper half space model, we consider the space $\{(s, x) \in \mathbb{R}^2 \times \mathbb{R}^+\}$ with the asymptotic boundary at $x = 0$. From [22] we have the following barrier result:

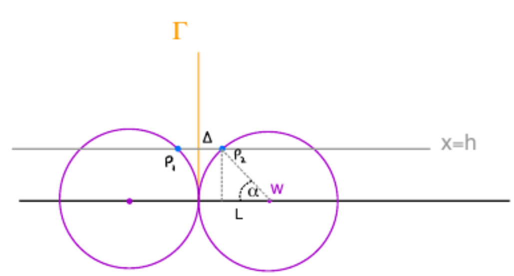
$$\text{if } s \in \mathbb{R}^2 \text{ and } 0 < r < d(x) = \operatorname{dist}(s, \partial Y), \text{ then } Y \cap B_r(s, 0) = \varnothing.$$

Here B_r is a Euclidean r-ball with center at $(s, 0)$. Let Γ be a vertical cylinder over γ:

$$\Gamma = \{(y, x) \in \mathbb{R}^2 \times \mathbb{R}^+ : y \in \gamma\}$$

Choose two smooth families of minimal hemispheres which lie to either side of Γ respectively and which are both tangent to γ. Letting $\Gamma^\pm$ be envelopes to these families, we get smooth mean-convex surfaces tangent to Γ along γ. It now suffices to consider any fixed $y \in \gamma$ for our calculations.

We can assume y lies at $(0, 0)$ in the closed half-plane containing y and intersecting Γ orthogonal. Fix w and consider the points $(w, 0)$ and $(-w, 0)$. For any positive h let p_1 and p_2 be the points where the line $x = h$ intersects the Euclidean w-balls centered at $(w, 0)$ and $(-w, 0)$. We calculate the horizontal distance $\triangle$ between p_1 and p_2.



Elementary trigonometry show that, for α the angle between the horizontal axis and the line segment from w to p_2, we get

$$\triangle = \frac{2h}{\tan\alpha}\left(\sqrt{\tan^2\alpha + 1} - 1\right)$$

In a sufficiently small neighborhood of the boundary, $\alpha << 1$ so we may use the binomial expansion to obtain

$$\triangle = \frac{2h}{\tan\alpha}\left(\frac{1}{2}\tan^2\alpha + \mathcal{O}(\tan^4\alpha)\right) \sim h\tan\alpha = h^2/L$$

where $\tan\alpha = h/L$ with $L \to w$ as $h \to 0$. This implies that any minimal graph function over Γ vanishes with order $\mathcal{O}(x^2)$. Now write $\widetilde{Y}$ as a normal graph over Γ via $\widetilde{u}$. Let ν be the unit normal vector field of Γ and η be the corresponding unit normal vector field for $\widetilde{Y}$. We see that $\widetilde{u}\nu + u\eta = \nu\mathcal{O}(x^2)$ and $|\eta(x) - \nu(x)| \to 0$ as $x \to 0$, giving the desired vanishing rate for u. □

Proposition 5.3. *Assume the initial setup. Suppose that $\{\Sigma_k\}$ converges*[2] *smoothly and graphically to either Σ_1 (or Σ_2). Then there exist a solution to the Jacobi equation on Σ_1 (or Σ_2).*

Proof. Without loss of generality, choose one of the limit points and denote by Σ_∞. Each Σ_k is associated with a smooth function u_k with compact domain $\Omega_k \subset\subset \Sigma_\infty$. We have $u_k(\partial\Omega_k) = 0$ by construction. Let ν_k and ν_∞ be unit normal vector fields of Σ_k and Σ_∞ respectively. Let $\eta \in C_c^\infty(\Omega_k)$ and $Z = \eta\nu_\infty$. Furthermore, denote $\Sigma_{k,t} = \text{graph}(tu_k)$ and $X = d(\exp_{\Sigma_\infty})\mid_{tu_k} u_k v_\infty$. We have

$$\begin{aligned} 0 &= H_{\Sigma_K}\nu_k \cdot Z = \text{div}_k Z \\ 0 &= H_{\Sigma_\infty}\nu_\infty \cdot Z = \text{div}_\infty Z \end{aligned}$$

for H the respective mean curvatures. Let $\{e_{k,t,i}\}$ be an orthonormal basis for $\Sigma_{k,t}$. First, we calculate

$$\begin{aligned} \frac{d}{dt}\left(\text{div}_{\Sigma_{k,t}} Z\right) = & \frac{d}{dt} g_t^{ij}\langle\nabla_{\partial i} Z, \partial j\rangle \\ = & - g_t^{ik} g_t^{jl}\dot{g}_{kl}\langle\nabla_{\partial i} Z, \partial j\rangle + g_t^{ij}\langle\nabla_{\partial t}\nabla_{\partial i} Z, \partial j\rangle + g_t^{ij}\langle\nabla_{\partial i} Z, \nabla_{\partial j}\partial t\rangle \\ = & - g_t^{ik} g_t^{jl}\left(2\langle\nabla_{\partial i}\partial t, \partial j\rangle\right)\langle\nabla_{\partial i} Z, \partial j\rangle \\ & + g_t^{ij}\langle\nabla_{\partial i}\nabla_{\partial t} Z + R^M(\partial t, \partial i)Z, \partial j\rangle + g_t^{ij}\langle\nabla_{\partial i} Z, \nabla_{\partial j}\partial t\rangle \\ = & - 2\langle\nabla_{e_i} X, e_j\rangle\langle\nabla_{e_i} Z, e_j\rangle \\ & + \langle\nabla_{e_i} Z, e_j\rangle\langle e_j, \nabla_{e_i} X\rangle + \langle\nabla_{e_i} Z, \nu_k\rangle\langle\nu_k, \nabla_{e_i} X\rangle \\ & + \langle\nabla_{e_i}\nabla_X Z, e_j\rangle + \langle R^M(X, e_i)Z, e_j\rangle \\ = & - \langle\nabla_{e_i} X, e_j\rangle\langle\nabla_{e_i} Z, e_j\rangle + \langle\nabla_{e_i} Z, \nu_k\rangle\langle\nu_k, \nabla_{e_i} X\rangle \\ & - \text{Ric}(X, Z) \end{aligned}$$

[2] Assume one sided convergence ($u_k > 0$). It remains to either justify or remove this assumption.

By the fundamental theorem of calculus, we have

$$\begin{aligned}\operatorname{div}_{\Sigma_k} Z - \operatorname{div}_{\Sigma_\infty} Z &= \int_0^1 \frac{d}{dt} \operatorname{div}_{\Sigma_{k,t}} Z dt \\ &= \int_0^1 - \operatorname{Ric}(X,Z)|_{\Sigma_{k,t}} - \langle \nabla_{e_i} Z, e_j\rangle \langle \nabla_{e_i} X, e_j\rangle|_{\Sigma_{k,t}} \\ &\quad + \langle \nabla_{e_i} Z, \nu_k\rangle\langle \nu_k, \nabla_{e_i} X\rangle\big|_{\Sigma_{k,t}}\end{aligned}$$

Also, $Z/\eta - \nu_{k,t} = \nu_\infty - \nu_{k,t} = \mathcal{O}(\nabla u_k)$ and likewise $X/u_k - \nu_{k,t} = \mathcal{O}(\nabla u_k)$. Calculating the inner products,

$$\begin{aligned}\langle \nabla e_i Z, e_j\rangle &= \langle \nabla_{e_i}(\nu_{k,t}\eta - \nu_{k,t}\eta + Z), e_j\rangle \\ &= \left(\langle \nabla_{e_i}\nu_{k,t}, e_j\rangle + \langle \nabla_{e_i}(Z/\eta - \nu_{k,t}), e_j\rangle\right)\eta \\ &= \left(A^{k,t}(e_i,e_j) + \mathcal{O}(\langle \nabla_{e_i}(\nabla u_k), e_j\rangle)\right)\eta \\ \langle \nabla_{e_i} X, e_j\rangle &= \left(A^{k,t}(e_i,e_j) + \mathcal{O}(\langle \nabla_{e_i}(\nabla u_k), e_j\rangle)\right)u_k \\ \langle \nabla_{e_i} Z, \nu_k\rangle &= \langle \nabla_{e_i}(\eta\nu_k), \nu_k\rangle + \langle \nabla_{e_i}\eta(\nu_\infty - \nu_k), \nu_k\rangle \\ &= \nabla_{\Sigma_{k,t}}\eta + \langle \nabla\eta \cdot \mathcal{O}(\nabla u_k), \nu_k\rangle \\ &= \nabla_{\Sigma_{k,t}}\eta + \left(\nabla\eta\cdot\mathcal{O}(\nabla u_k) + \eta\mathcal{O}(\nabla^2 u_k)\right) \\ \langle \nabla_{e_i} X, \nu_k\rangle &= \nabla_{\Sigma_{k,t}} u_k + \left(\nabla u_k \cdot \nabla u_k + u_k \cdot \nabla^2 u_k\right)\end{aligned}$$

We now examine the difference in mean curvature at each point.

$$\begin{aligned}H_{\Sigma_k} - H_{\Sigma_\infty} &= \frac{\operatorname{div}_{\Sigma_k} Z}{\nu_k \cdot Z} - \frac{\operatorname{div}_{\Sigma_\infty} Z}{\eta} + \frac{\operatorname{div}_{\Sigma_k} Z}{\eta} - \frac{\operatorname{div}_{\Sigma_k} Z}{\eta} \\ &= \left(\frac{1}{Z\cdot\nu_k} - \frac{1}{\eta}\right)\operatorname{div}_{\Sigma_k} Z + \frac{1}{\eta}\left(\operatorname{div}_{\Sigma_k} Z - \operatorname{div}_{\Sigma_\infty} Z\right) \\ &= H_{\Sigma_k}(1 - \nu_k\cdot\nu_\infty) + \frac{1}{\eta}\left(\operatorname{div}_{\Sigma_k} Z - \operatorname{div}_{\Sigma_\infty} Z\right)\end{aligned}$$

Thus, we note that the first term vanishes in the limit $k \to \infty$. Plugging in and integrating over the domain in the surface Σ_∞ for which u_k is defined, we obtain

$$\begin{aligned}0 &= \int_{\Omega_k} (H_{\Sigma_k} - H_{\Sigma_\infty})\eta = \int_{\Omega_k} H_{\Sigma_k}(1 - \nu_k\cdot\nu_\infty)\eta \\ &+ \int_0^1 \int_{\Omega_k} - \operatorname{Ric}(X,Z)|_{\Sigma_{k,t}} - \left|A^{k,t}\right|^2 \eta u_k + \nabla_{\Sigma_{k,t}}\eta \cdot \nabla_{\Sigma_{k,t}} u_k + (\dots)\end{aligned}$$

Integrating by parts to remove the ∇ from η,

$$\begin{aligned}&= \int_{\Omega_k} H_{\Sigma_k}(1 - \nu_k\cdot\nu_\infty)\eta \\ &+ \int_0^1 \int_{\Omega_k} - \operatorname{Ric}(\nu_\infty,\nu_\infty)\eta u_k - \left|A^{k,t}\right|^2 \eta u_k - \eta \operatorname{div}(\nabla_{\Sigma_{k,t}} u_k) + (\dots)\end{aligned}$$

The term in parenthesis is given explicitly by

$$
\begin{aligned}
(\dots) = &-2\eta u_k A(e_i, e_i)\mathcal{O}(\langle \nabla_{e_i}(\nabla u_k), e_j\rangle) - \eta u_k \mathcal{O}(\langle \nabla_{e_i}(\nabla u_k), e_j\rangle) \\
&+ \nabla_{\Sigma_{k,t}}\eta(\nabla u_k \cdot \nabla u_k + u_k \cdot \nabla^2 u_k) + \nabla_{\Sigma_{k,t}} u_k(\nabla\eta \cdot \mathcal{O}(\nabla u_k) + \eta\mathcal{O}(\nabla^2 u_k)) \\
&+ (\nabla\eta \cdot \mathcal{O}(\nabla u_k) + \eta\mathcal{O}(\nabla^2 u_k))(\nabla u_k \cdot \nabla u_k + u_k \cdot \nabla^2 u_k)
\end{aligned}
$$

and vanishes in the limit k to infinity. Renormalize so that $\widetilde{u}_k = u_k \|u_k\|^{-1}_{L^2(\Omega_k)}$. By our initial assumption, we have that $\widetilde{u}_k$'s are all positive. These $\widetilde{u}_k$'s are positive solutions to a sequence of uniformly elliptic equations with smooth coefficients, where $L_k \to L$, the Jacobi operator. Fix $0 < \varepsilon \ll \frac{1}{4}R_{\min}$ and let K be the set

$$K = \Sigma_\infty \cap \{x \geq \varepsilon\}$$

so that there exists a k_0 such that for $k > k_0$, we have $K \subset\subset \Omega_k$. We have $L_k \widetilde{u}_k = 0$ on Ω_k so eventually we get

$$\sup_K \widetilde{u}_k \leq C_k \inf_K \widetilde{u}_k$$

by Harnack estimates. Here $C_k = C(k)$ only depends on the ellipticity and boundedness of the L_k coefficients since K is fixed. However since $L_k \to L$ smoothly, the coefficients of L_k are close and we can find a (largest) uniform constant C such that

$$\sup_K \widetilde{u}_k \leq C \inf_K \widetilde{u}_k$$

Furthermore, on $\Omega_k \setminus K$ we have the necessary bounds and decay of $\widetilde{u}_k$ by Lemma 5.4 and Lemma 5.5 for each k. This gives the decay of the limiting function on $\Sigma_\infty \setminus K$. Thus (since $\widetilde{u}_k > 0$) we can conclude that $\widetilde{u}_k$ converges locally and smoothly to a non-trivial solution $\widetilde{u} : \Sigma_\infty \to \mathbb{R}_+$ of

$$-\Delta_{\Sigma_\infty}\widetilde{u} - (|A|^2 + \mathrm{Ric}(\nu_\infty, \nu_\infty))\widetilde{u} = 0$$

that vanishes as we approach the ideal boundary γ.

□

Lemma 5.4. *If γ has finite curvature at all points then $\frac{d}{dx}u_k(\partial\Omega_k)$ is bounded for sufficiently large k.*

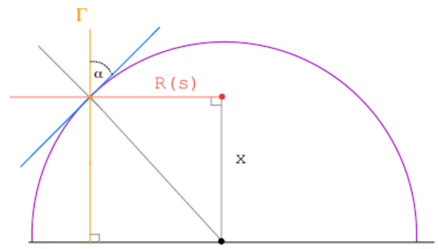
Proof of lemma. First assume that $\gamma_k = (\gamma(s), \frac{1}{k})$. That is, γ_k is the intersection of the plane $x = \frac{1}{k}$ with the vertical cylinder Γ over γ. Then the curvature $\kappa(s)$ of γ and γ_k coincide at all points s. The radius of any osculating circle at a point $s \in \gamma$ is given by

$$R(s) = \frac{1}{|\kappa(s)|}$$

Such an osculating circle coincides with the intersection of the plane $x = \frac{1}{k}$ with some minimal hemisphere centered at a point on $x = 0$, for sufficiently large k. Letting

$$\kappa_{\max} = \sup_{s\in\gamma} \kappa(s)$$

we get that $R_{\min} = \kappa_{\max}^{-1}$ is the radius of the smallest circle that is tangent to all points $s \in \gamma$. Via maximum principle, we bound the derivative at γ_k with barriers being minimal hemispheres such that their intersection with the plane $x = \frac{1}{k}$ coincides with the osculating circle at that height.



Denoting α to be the angle between Γ and the tangent along each barrier at height x, elementary geometry shows that $\alpha(s) = \tan^{-1}(x|\kappa(s)|)$. Hence at fixed height x,

$$\alpha_{\max} = \tan^{-1}(x|\kappa_{\max}|)$$

is the uniform max angle achieved among all points of γ. We see then that graph functions u_k on $\Gamma \cap \{x \geq \frac{1}{k}\}$ with $u_k(x = \frac{1}{k}) = 0$ and must locally obey

$$\left|\frac{d}{dx} u_k(\partial\Omega_k)\right| \leq \frac{d}{dx} f_k\left(1/k\right)$$

where f_k over Γ is given by

$$f_k(x') = \begin{cases} 0 & \text{if } x' < \frac{1}{k} \\ \frac{1}{k}|\kappa_{\max}|x' & \text{if } x' \geq \frac{1}{k} \end{cases}$$

Now for $\gamma_k = \partial\Omega_k$ obtained by our initial setup, we have

$$|(\gamma_k(s), x) - (\gamma(s), x)| < \mathcal{O}(x^2)$$

by Lemma 5.2 for all s. Smooth convergence $\gamma_k \to \gamma$ gives

$$|\kappa_k(s) - \kappa(s)| < \varepsilon$$

for sufficiently small x. Consider $\Gamma_k = \{(\gamma_k, x) \mid x \in \mathbb{R}_{\geq 0}\}$ the vertical cylinder of γ_k. Repeating the above argument gives an angle $\alpha_{\max,k} = \tan^{-1}(x|\kappa_{max,k}|)$ with respect to Γ_k that acts as bounds on both sides of Γ_k in the following sense: $T\Sigma_\infty$ will necessarily make an angle β_k with respect to Γ_k at the height $x_{0,k}$ of $\Gamma_k \cap \Sigma_\infty$ and we know that $\beta_k \to 0$ by Lemma 5.2. Then since u_k is defined over $\Omega_k \subset \Sigma_\infty$ the angles bounding $\left|\frac{d}{dx} u_k(\partial\Omega_k)\right|$ are $\alpha_{\max,k} \pm \beta_k$ respectively on each side. At this point the rest of the above argument applies with the bounding functions

$$f_k(x')^{\pm} = \begin{cases} 0 & \text{if } x' < x_{0,k} \\ \tan(\alpha_{\max,k} \pm \beta_k)x' & \text{if } x' \geq x_{0,k} \end{cases}$$

where $\alpha_{\max,k} \to \alpha_{\max}$.

$\square$

Lemma 5.5. *Let $\theta(x, \varepsilon)$ the horizontal strip bounded by x and $x+\varepsilon$ and denote $Y(x, \varepsilon) := Y \cap \theta(x, \varepsilon)$. For $x < \frac{1}{4}R_{\min}$ there exists an ε independent of x and constant C such that*

$$\int_{\Sigma_\infty(x,\varepsilon)} |u_k|^2 < C$$

That is, there is control of u_k in a collar ε-neighborhood of $\partial\Omega_k$

Proof. An osculating circle of radius $R_{\min}$ at height x corresponds to a minimal hemisphere of radius $\sqrt{R_{\min}^2 + x^2}$ centered on $(0, R_{\min})$. At a height $x < R_{\min}$, consider the osculating circle $R(s) \geq R_{\min}$ for all $s \in \gamma$. Such a circle corresponds to a minimal hemisphere of radius $\overline{R}$ satisfying

$$\overline{R} = \sqrt{R(s)^2 + x^2} \geq \sqrt{R_{\min}^2 + x^2} > \sqrt{2}x > x$$

Thus letting $\varepsilon = \frac{(\sqrt{2}-1)}{2}R_{\min}$, we note that

$$\begin{aligned} R_{\min} + \varepsilon &= \frac{1+\sqrt{2}}{2}R_{\min} < \sqrt{2}R_{\min} \\ &\leq \sqrt{R_{\min}^2 + R^2(s)} \end{aligned}$$

which is the radius of the minimal hemisphere corresponding to $R(s)$ and intersecting Γ at height $R_{\min}$. Hence for arbitrary $s \in \gamma$ and $x < R_{\min}$, we know there exists a minimal hemisphere intersecting $\theta(x, \varepsilon)$ and coinciding with the osculating circle at s. Given a graph function g_k over Γ with $\text{supp}(g_k) = \Omega_k \subset \Gamma$, we then have

$$\int_{\Gamma(x,\varepsilon)} |g_k|^2 \leq \int_{\Gamma(x,\varepsilon)} |h(x)|^2 \leq \int_{\Gamma(x,\varepsilon)} \|h\|_\infty^2 = \|h\|_\infty^2 m(\Gamma(x, \varepsilon))$$

where $h(x') = -\sqrt{R(s)^2 + x^2 - x'^2} + R(s)$ parameterizes the minimal hemisphere over Γ. Taking roots gives us

$$\|g_k\|_{L^2(\Gamma(x,\varepsilon))} \leq \sqrt{m(\Gamma(x, \varepsilon))}\|h\|_\infty =: C$$

It remains to extend for u_k over Σ_∞. In order to use the above argument we require that the ray along the unit normal vector field $\nu_\infty(x)$ at height x of Σ_∞ intersects the minimal hemisphere over Γ. Let $f(x) = \sqrt{R_{\min}^2 - x^2} - R_{\min}$ parameterize the minimal $R_{\min}$-hemisphere over Γ intersecting at γ. For any a, we have the tangent at height a given by

$$L(x) = f(a) + f'(a) - af'(a)$$

letting $b = af'(a) + f(a)$ and $m = f'(a)$, the perpendicular to $L(x)$ at a is given by

$$L^{\perp}(x) = f(a) + \frac{a}{f'(a)} - \frac{1}{f'(a)}x$$

Letting $a = \frac{1}{4}R_{\min}$, $a + \varepsilon = \frac{2\sqrt{2}-1}{4}R_{\min}$ and we have that $L^{\perp}$ at $a + \epsilon$ intersects the hemisphere whenever

$$f(a+\epsilon) + \frac{a+\epsilon}{f'(a+\epsilon)} - \frac{1}{f'(a+\epsilon)}x = -\sqrt{R_{\min}^2 - x^2} + R_{\min}$$

solving for x, we indeed get two solutions

$$x = R_{\min}\left(\frac{2\sqrt{2}-1}{4}\right)\left(2\sqrt{1 - \frac{9-4\sqrt{2}}{16}} \pm \sqrt{1 - \frac{9-4\sqrt{2}}{4}}\right)$$

so for $x < \frac{1}{4}R_{\min}$, the ray along $\nu_\infty(x)$ on $x \in (x, x+\varepsilon)$ intersects the minimal hemisphere. Hence by the above argument we get

$$\int_{\Sigma_\infty(x,\varepsilon)} |u_k|^2 < \sqrt{m(\Sigma_\infty(x,\varepsilon))}\|h\|_\infty < \infty$$

where h is some (bounded) function parameterizing the portion of the $R_{\min}$-hemisphere intersecting $\theta(x,\varepsilon)$ and over Σ_∞. □

For $M = \mathbb{H}^3$, denote by $\mathcal{M}_k(M)$ the space of all properly embedded minimal surfaces of genus k which extend to $\overline{M}$ as $\mathcal{C}^{3,\alpha}$ submanifolds with boundary and which intersects ∂M orthogonally. Denote the space $\mathcal{E}$ by all $\mathcal{C}^{3,\alpha}$ closed embedded curves $\gamma \subset \partial M$. Define the map

$$\Pi : \mathcal{M}_k(M) \to \mathcal{E}(\partial M)$$

by $\Pi(Y) = \partial Y$. The results of [3] give that Π is map between Banach manifolds, and furthermore that it is Fredholm with index 0. As explained in [23], replace $\mathcal{C}^{k,\alpha}$ by its closure in $\mathcal{C}^\infty$. Under this regularity restriction both of the above are separable Banach manifolds.

Lemma 5.6. *Denote $\Sigma \in \mathcal{M}_k(M)$. For generic boundaries $\gamma \in \mathcal{E}(\partial M)$, there exists no nontrivial Jacobi fields on Σ fixing γ.*

Proof. By Sard-Smale [24], the regular values of Π are generic in $\mathcal{E}(\partial M)$. That is:

$$\gamma \in \mathcal{E}(\partial M) \text{ such that } \Sigma \in \Pi^{-1}(\gamma) \implies D\Pi_\Sigma \text{ surjective} \iff \text{coker} D\Pi_\Sigma = \varnothing$$

Fix $\Sigma = \Sigma_0$ where Σ_t is a curve in $\mathcal{M}_k(M)$ corresponding to a 1-parameter family of minimal surfaces in M with $\gamma_t = \partial\Sigma_t$ a curve in $\mathcal{E}(\partial M)$. Denote ϕ_t tangent vectors to the curve Σ_t. The map

$$D\Pi : T_\Sigma \mathcal{M}_k(M) \to T_\gamma \mathcal{E}(\partial M)$$

sends ϕ_t to tangent vectors of γ_t in $\mathcal{E}(\partial M)$. Hence we get that

$$\ker D\Pi = \{\phi_t \in T_\Sigma \mathcal{M}_k(M) : D\Pi(\phi_t) = 0\}$$

identifies with the space of Jacobi fields that that fix γ. Finally, Proposition 4.2 of [3] gives

$$\dim \ker D\Pi = \dim \operatorname{coker} D\Pi$$

so by surjectivity of $D\Pi$ for generic curves we get $\dim \ker D\Pi = 0$. Hence there are no decaying Jacobi fields on Σ. □

The desired result follows now from the previous results.

Proposition 5.7. *Σ is distinct from Σ_1 and Σ_2.*

Proof. Suppose Σ_k converges to Σ_1 or Σ_2. By Proposition 5.3 there exists a nontrivial vanishing Jacobi field on Σ, contradicting Lemma 5.6. □

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